Sensitive dependence on initial conditions in maps

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Lyapunov exponents

- Sensitive dependence on initial conditions is a measurable quantity using a quantity called the Lyapunov exponent.
- Locally, on average, nearby points diverge exponentially if the map is chaotic.
- Suppose that we have two sequences generated by the same map from nearby initial conditions, $\{x_n\}$ and $\{y_n\}$.
  Define $\delta_n = y_n - x_n$.
- On average, the distance grows exponentially with iteration:

  $$|\delta_n| \sim |\delta_0| e^{\mu n}$$

  μ is the Lyapunov exponent.
- Since μ is an average quantity, we need long sequences to estimate it accurately.

Estimating the Lyapunov exponent of a map

- Starting from $|\delta_n| \sim |\delta_0| e^{\mu n}$, we can obtain

$$\mu \sim \frac{1}{n} \ln \left| \frac{\delta_n}{\delta_0} \right|$$

- The latter is not exact anywhere. The '$\sim$' symbol here indicates that it is true on average over the attractor.
- Since $y_0 = x_0 + \delta_0$,

$$\delta_n = y_n - x_n = f^{(n)}(x_0 + \delta_0) - f^{(n)}(x_0)$$

$$\therefore \mu \sim \frac{1}{n} \ln \left| \frac{f^{(n)}(x_0 + \delta_0) - f^{(n)}(x_0)}{\delta_0} \right|$$

- For very small δ_0, this is

$$\mu \sim \frac{1}{n} \ln \left| \frac{df^{(n)}}{dx} \right|_{x=x_0}$$

Estimating the Lyapunov exponent of a map
(continued)

$$\mu \sim \frac{1}{n} \ln \left| \frac{df^{(n)}}{dx} \right|_{x=x_0}$$

- $f^{(n)}(x) = f(f(f(\ldots f(x)))\ldots)$
- Apply the chain rule to the derivative:

$$\begin{aligned}
\left.\frac{df^{(n)}}{dx}\right|_{x=x_0} &= \left.\frac{df}{dx}\right|_{x=f^{(n-1)}(x_0)} \left.\frac{df}{dx}\right|_{x=f^{(n-2)}(x_0)} \cdots \left.\frac{df}{dx}\right|_{x=x_0} \\
&= \left.\frac{df}{dx}\right|_{x=x_{n-1}} \left.\frac{df}{dx}\right|_{x=x_{n-2}} \cdots \left.\frac{df}{dx}\right|_{x=x_0} \\
&= \prod_{i=0}^{n-1} f'(x_i).
\end{aligned}$$

Estimating the Lyapunov exponent of a map
(continued)

$$\mu \sim \frac{1}{n} \ln \left| \frac{df^{(n)}}{dx} \right|_{x=x_0} \qquad \text{and} \qquad \left. \frac{df^{(n)}}{dx} \right|_{x=x_0} = \prod_{i=0}^{n-1} f'(x_i).$$

- Substitute into μ and use the product rule of logarithms:

$$\mu \sim \frac{1}{n} \ln \left| \prod_{i=0}^{n-1} f'(x_i) \right| = \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|$$

- To get an accurate average, we need to include a large number of observations, so formally

$$\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|$$

Some simple cases of Lyapunov exponents

Stable fixed points

$$\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|$$

- Suppose a map has a stable fixed point x^*.
  - For any sequence started from within the basin of attraction of x^*, because trajectories approach x^*, the average is eventually dominated by the fixed point, so
$$\mu = \ln |f'(x^*)|$$
  - Since $|f'(x^*)| < 1$ for a stable fixed point, $\mu < 0$.

Some simple cases of Lyapunov exponents

Stable fixed points

$$\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|$$

- Suppose a map has a stable period-k orbit.
  - The same argument applies as above, except that we now have to average over all k points of the periodic solution:

$$\mu = \frac{1}{k} \sum_{i=1}^{k} \ln |f'(x_i)|$$

    where $(x_1, x_2, \ldots, x_k)$ are the k points of the stable periodic orbit.
  - Note that

$$\mu = \frac{1}{k} \ln \left| \frac{df^{(k)}}{dx} \right|_{x=x_1}$$

  - Since the orbit is stable, this derivative must have a magnitude smaller than one, so $\mu < 0$.

Matlab one-liners

- Matlab functions and expressions can often operate elementwise on arrays.
  This facilitates the writing of one-liners.
- In the following, suppose that `x` is a one-dimensional array.
- The expression `lambda*(1-2*x)` multiplies each element of the array by 2, then subtracts each element from 1, and finally multiplies each element by `lambda`. The result is an array storing the result of applying the above formula to each element of `x`.
- `abs(x)` applies the absolute value function to each element of `x`.
  Most Matlab functions of a single variable (`log`, `sin`, ...) will operate elementwise on arrays.

A Matlab/Octave one-liner for the Lyapunov exponent

$$\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|$$

- Suppose that we have evaluated iterates of a map and stored them in an array `x`.
- Suppose that the function `dfdx` evaluates $f'(x)$, applying elementwise to an array `x`.
- Then the Lyapunov exponent can be estimated from the following one-liner:

```
mu = mean(log(abs(dfdx(x))))
```