Stability and bifurcation analysis for maps

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Fixed points

- An equilibrium point is a point that doesn't change under the action of the time evolution operator.
- Equilibrium points of maps are usually called fixed points.
- A fixed point of a map $\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n, \mathbf{x}_{n-1}, \mathbf{x}_{n-2}, \ldots)$ satisfies

$$\mathbf{x}^* = \mathbf{f}(\mathbf{x}^*, \mathbf{x}^*, \mathbf{x}^*, \ldots)$$

Example: fixed points of the logistic map

- Consider the logistic map

$$x_{n+1} = \lambda x_n(1 - x_n)$$

- The fixed points of this map satisfy

$$x = \lambda x(1 - x)$$
$$\therefore x\left[1 - \lambda(1 - x)\right] = 0$$
$$\therefore x^\dagger = 0 \qquad \text{or} \qquad x^* = 1 - \frac{1}{\lambda}$$

Stability of fixed points

- Suppose x^* is a fixed point of the map $x_{n+1} = f(x_n)$.
- Consider a small perturbation from the fixed point such that $x_n = x^* + \delta x_n$.
- Then

$$\begin{aligned} x_{n+1} &= f(x_n) = f(x^* + \delta x_n) \\ &\approx f(x^*) + f'(x^*)\delta x_n \\ &= x^* + f'(x^*)\delta x_n. \end{aligned}$$

  But

$$x_{n+1} = x^* + \delta x_{n+1}.$$

  Comparing the two expressions for x_{n+1}, we get

$$\delta x_{n+1} \approx f'(x^*)\delta x_n.$$

Stability of fixed points (continued)

So far: $\delta x_{n+1} \approx f'(x^*)\delta x_n$

Theorem: A fixed point x^* of a map $x_{n+1} = f(x_n)$ is

- stable if $|f'(x^*)| < 1$, or
- unstable if $|f'(x^*)| > 1$.

Stability of the fixed points of the logistic map

$$f(x) = \lambda x(1 - x)$$

$$\therefore f'(x) = \lambda(1 - 2x)$$

For $x^\dagger = 0$: $f'(0) = \lambda$.
This fixed point is stable if $\lambda < 1$, and unstable if $\lambda > 1$.

For $x^* = 1 - \frac{1}{\lambda}$: $f'(x^*) = \lambda\left[1 - 2\left(1 - \frac{1}{\lambda}\right)\right] = 2 - \lambda$.

- This fixed point is stable if $|2 - \lambda| < 1$.
- This is equivalent to $-1 < 2 - \lambda < 1$, or $1 < \lambda < 3$.

Periodic orbits

- A period-k periodic orbit is a sequence of iterates of the map such that

$$x_1 \to x_2 \to x_3 \to \dots x_k \to x_1$$

- For a period-2 orbit, $x_2 = f(x_1)$, and $x_1 = f(x_2)$.
- It follows that $x_1 = f(x_2) = f(f(x_1))$.
- In other words, period-2 orbits are fixed points of the map $x_{n+2} = f(f(x_n))$.
- In general, a period-k orbit is a fixed point of $x_{n+k} = f^{(k)}(x_n)$.
- The stability of periodic orbits can be studied by studying the stability of the map composed with itself k times.

Example: periodic orbits of the logistic map

- This is a good opportunity to use Maple. . .

Period-3 implies chaos

- If a map has a period-3 orbit, then it is a theorem that it must have orbits of all periods (Li and Yorke, 1975).
- Moreover, a period-three orbit implies that there exist points x_0 and y_0 such that applying the map starting from each of these points generates sequences in which the points pass arbitrarily close to each other, then move apart again, and later again pass arbitrarily close to each other, then move part, ... at infinitum.
- These properties guarantee a chaotic trajectory, though not necessarily a chaotic attractor.
- However, in practice, you often (generally?) find a chaotic attractor "nearby" in parameter space if you find a period-3 orbit.