Hamiltonian systems

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Hamiltonian systems

- Suppose that we have a dynamical system whose equations of motion are related to a function $H(\mathbf{x}, \mathbf{p})$ by

$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial x_i}$$

- Such a system is called a Hamiltonian system.
- $H(\mathbf{x}, \mathbf{p})$ is called the Hamiltonian function, or just the Hamiltonian.
- If the vectors $\mathbf{x}$ and $\mathbf{p}$ are elements of $\mathbb{R}^n$, then we say that n is the number of degrees of freedom.
  The phase space is $2n$-dimensional.

Significance of the Hamiltonian

- The Hamiltonian is a conserved quantity.
- To see this, differentiate $H(\mathbf{x}, \mathbf{p})$ with respect to time using the chain rule:

$$\begin{aligned}\frac{dH}{dt} &= \sum_{i=1}^{n} \frac{\partial H}{\partial x_i}\,\frac{dx_i}{dt} + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\,\frac{dp_i}{dt} \\ &= \sum_{i=1}^{n} \frac{\partial H}{\partial x_i}\frac{\partial H}{\partial p_i} + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\left(-\frac{\partial H}{\partial x_i}\right) \\ &= 0\end{aligned}$$

Conservative mechanical systems are Hamiltonian

Example: harmonic oscillator

- Consider a harmonic oscillator with Hooke's law force

$$F = -kx$$

- Since

$$F = \frac{dp}{dt}$$

(most general form of $F = ma$)

$$\frac{dp}{dt} = -kx$$

- Also,

$$\frac{dx}{dt} = v = p/m$$

Conservative mechanical systems are Hamiltonian

Example: harmonic oscillator (continued)

$$\frac{dx}{dt} = p/m \qquad \frac{dp}{dt} = -kx$$

- If this system is Hamiltonian, then we must have

$$\frac{\partial H}{\partial p} = p/m \qquad \text{and} \qquad \frac{\partial H}{\partial x} = kx$$

Conservative mechanical systems are Hamiltonian

Example: harmonic oscillator (continued)

- From $\frac{\partial H}{\partial p} = p/m$, we get

$$H = \frac{p^2}{2m} + f(x)$$

- And from $\frac{\partial H}{\partial x} = kx$,

$$H = \frac{1}{2}kx^2 + g(p)$$

- Therefore

$$H = \frac{1}{2}kx^2 + \frac{p^2}{2m}$$

Dissipative mechanical systems are not Hamiltonian

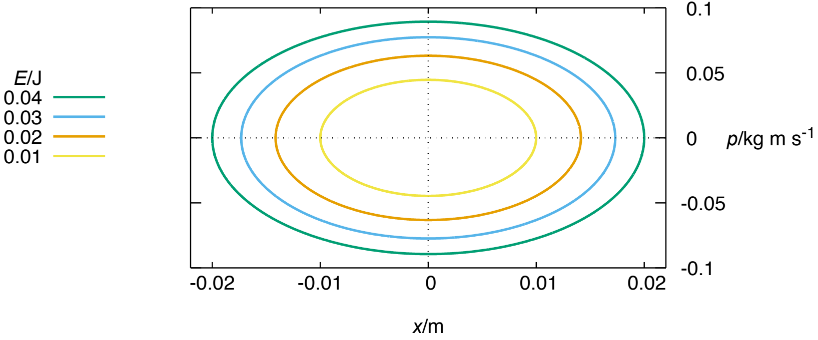
- If we take

$$F = \frac{dp}{dt} = -kx - \mu v$$

  which includes a frictional (dissipative) term $-\mu v$, we get a system that is not Hamiltonian.
  Try it!

Hamiltonian systems in two dimensions

- For a two-dimensional Hamiltonian system, the Hamiltonian $H(x, p)$ defines solution curves in phase space.
- Example: harmonic oscillator with $k = 200\,\mathrm{N\,m^{-1}}$ and $m = 0.1\,\mathrm{kg}$



Integrability

- The Poisson bracket of two functions $H(\mathbf{x}, \mathbf{p})$ and $L(\mathbf{x}, \mathbf{p})$ is defined by

$$\{H, L\} = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p_i} \frac{\partial L}{\partial x_i} - \frac{\partial H}{\partial x_i} \frac{\partial L}{\partial P_i} \right)$$

  differentiable functions.
- $L(\mathbf{x}, \mathbf{p})$ is a first integral of a Hamiltonian system if $\dot{L} = 0$.
- $L(\mathbf{x}, \mathbf{p})$ is a first integral of a system with Hamiltonian $H(\mathbf{x}, \mathbf{p})$ if $\{H, L\} = 0$.
- A Hamiltonian system with n first integrals is completely integrable.

An example: particle in a two-dimensional harmonic trap

- A Paul trap holds ions in a well-defined region of space using electric fields.
  Overall, the potential is harmonic and can have a spherical geometry, or hold the ions in a relatively flat disk.
- Hamiltonian for a two-dimensional ion trap:

$$H = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2}k(x^2 + y^2)$$

- Since there is no external torque, the angular momentum $L = xp_y - yp_x$ should be a constant of the motion.
  (Check this.)

Particle in a two-dimensional harmonic trap (continued)

- We have two degrees of freedom and two first integrals (H and L) so this system is completely integrable.
- The trajectories lie at the intersection of $H(x, y, p_x, p_y) = E$ and $L(x, y, p_x, p_y) = \ell$.
- This intersection is a two-dimensional surface in the four-dimensional phase space.