Singularly perturbed systems and Tikhonov's theorem

Marc R. Roussel

October 16, 2019

Perturbation problems

- Suppose that we have a set of equations containing two small parameters, ϵ and δ:

$$\frac{dx}{dt} = f(x, y) + \delta g(x, y)$$
$$\epsilon \frac{dy}{dt} = h(x, y)$$

- If setting one of these parameters to zero gives us a problem we can solve, we have a perturbation problem, in which we try to use the known solution as a starting point for developing an approximate solution for small values of ϵ or δ.

Regular and singular perturbations

$$\frac{dx}{dt} = f(x,y) + \delta g(x,y)$$
$$\epsilon \frac{dy}{dt} = h(x,y)$$

- Setting $\delta = 0$ does not change the nature of the equations: We still have two differential equations. We call the perturbation problem associated with δ a regular perturbation problem.
- Setting $\epsilon = 0$ changes the nature of the equations as the second differential equation is replaced by the algebraic equation $h(x,y) = 0$. We call the perturbation problem associated with ϵ a singular perturbation problem.

Systems associated with a singular system

Given a system in singular perturbation form

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, \mathbf{z}),$$
$$\epsilon \frac{d\mathbf{z}}{dt} = \mathbf{g}(\mathbf{x}, \mathbf{z})$$

we define

- The degenerate system

  $$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, \mathbf{z}),$$
  $$\mathbf{z} = \phi(\mathbf{x})$$

  where $\mathbf{z} = \phi(\mathbf{x})$ is the solution of the equation $\mathbf{g}(\mathbf{x}, \mathbf{z}) = \mathbf{0}$.
- The adjoined system

  $$\frac{d\mathbf{z}}{dt} = \mathbf{g}(\mathbf{x}, \mathbf{z})$$

  in which $\mathbf{x}$ is treated as a constant.

Tikhonov's theorem

When $\epsilon \to 0$, the solution of the system

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, \mathbf{z}),$$
$$\epsilon\frac{d\mathbf{z}}{dt} = \mathbf{g}(\mathbf{x}, \mathbf{z})$$

tends to the solution of the corresponding degenerate system provided $\mathbf{z} = \phi(\mathbf{x})$ is a stable solution of the adjoined system.