Chemistry 4010 Lecture 2: Linear stability analysis for ODEs

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Linearizing an ODE

- Suppose we have a set of ODEs

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

  with an equilibrium point $\mathbf{x}^*$, i.e. $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$.
  How do the trajectories near $\mathbf{x}^*$ behave?
- Taylor expansion of $\mathbf{f}$ near $\mathbf{x}^*$:

$$\dot{\mathbf{x}} = \cancelto{0}{\mathbf{f}(\mathbf{x}^*)} + \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}^*} (\mathbf{x} - \mathbf{x}^*) + O\left((\mathbf{x} - \mathbf{x}^*)^2\right)$$

- Big O notation: denotes terms with an exponent at least as large, in this case terms with an exponent of at least 2.
- If $\mathbf{x}$ is sufficiently close to $\mathbf{x}^*$, then

$$\dot{\mathbf{x}} \approx \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}^*} (\mathbf{x} - \mathbf{x}^*)$$

Linearizing an ODE

- $\partial \mathbf{f}/\partial \mathbf{x}$ is the Jacobian matrix, often denoted $\mathbf{J}$:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}.$$

- We need the Jacobian evaluated at $\mathbf{x}^*$, which is a constant matrix $\mathbf{J}^*$. The linearization becomes

$$\dot{\mathbf{x}} = \mathbf{J}^*(\mathbf{x} - \mathbf{x}^*)$$

- ... or, if we define $\delta\mathbf{x} = \mathbf{x} - \mathbf{x}^*$,

$$\boxed{\dot{\delta\mathbf{x}} = \mathbf{J}^*\,\delta\mathbf{x}}$$

Linearizing an ODE

- $\dot{\delta\mathbf{x}} = \mathbf{J}^* \,\delta\mathbf{x}$ is a linear ordinary differential equation.
- The solution can be written

$$\delta\mathbf{x} = \sum_{j=1}^{n} a_j \mathbf{e}_j e^{\lambda_j t}$$

where $(\lambda_j, \mathbf{e}_j)$ is one of the n eigenvalue-eigenvector pairs of the matrix $\mathbf{J}^*$, and the a_j are constants chosen to satisfy the initial conditions.

Theorem: The equilibrium point $\mathbf{x}^*$ is locally stable if all of the λ_j have negative real parts.
It is unstable if at least one of the λ_j has a positive real part.

Technical details: eigenvalues of a matrix

- The eigenvalues λ of a matrix $\mathbf{A}$ satisfy the characteristic equation

$$|\lambda\mathbf{I} - \mathbf{A}| = 0$$

- The left-hand side of the characteristic equation is a polynomial in λ.
- If $\mathbf{A}$ is an $n \times n$ matrix, then the characteristic polynomial has degree n, so there are n eigenvalues.
- The eigenvalues can be real, or can appear in complex-conjugate pairs $\lambda_{j,j+1} = \mu_j \pm i\nu_j$.

Technical details: exponentials of complex numbers

- The solutions of a linear differential equation involve the exponentials of the eigenvalues, $e^{\lambda_j t}$.
- If $\lambda_j = \mu_j + i\nu_j$, then
$$e^{\lambda_j t} = e^{\mu_j t} e^{i\nu_j t}$$
- Euler's formula allows us to expand the exponential with the imaginary argument:
$$e^{i\nu_j t} = \cos(\nu_j t) + i \sin(\nu_j t)$$

Conclusion: From the point of view of stability, only the real part of the eigenvalue matters.

A quick recap

- The behavior near the equilibrium point is governed by the differential equation $\dot{\delta \mathbf{x}} = \mathbf{J}^* \, \delta \mathbf{x}$ with solution

$$\delta \mathbf{x} = \sum_{j=1}^{n} a_j \mathbf{e}_j e^{\lambda_j t}$$

- The linear stability theorem says the following:

  The equilibrium point $\mathbf{x}^*$ is locally stable if all of the λ_j have negative real parts.

  It is unstable if at least one of the λ_j has a positive real part.

- Note that the theorem says nothing about what happens if there are eigenvalues with zero real part, the rest all having negative real parts.