A model with an Andronov-Hopf bifurcation

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The following is a highly simplified dimensionless model of the reaction catalyzed by phosphofructokinase [1]:

$$\begin{aligned}\dot{x} &= 1 - xy^{\gamma},\\ \dot{y} &= \alpha(xy^{\gamma} - y).\end{aligned}$$

In these equations, x is a dimensionless variable for the substrate concentration, and y represents the product concentration. The reaction is product-activated, which implies that the parameter $\gamma > 1$.

It is easy to show that this system has a single equilibrium point at $(x^*, y^*) = (1, 1)$.

The Jacobian is

$$\mathbf{J} = \begin{bmatrix} -y^{\gamma} & -\gamma xy^{\gamma-1} \\ \alpha y^{\gamma} & \alpha(\gamma xy^{\gamma-1} - 1) \end{bmatrix}$$

At the equilibrium point,

$$\mathbf{J}^* = \begin{bmatrix} -1 & -\gamma \\ \alpha & \alpha(\gamma - 1) \end{bmatrix}$$

The characteristic equation is found by

$$\begin{aligned}|\lambda\mathbf{I} - \mathbf{J}^*| &= \begin{vmatrix} \lambda + 1 & \gamma \\ -\alpha & \lambda - \alpha(\gamma - 1) \end{vmatrix}\\ &= (\lambda + 1)[\lambda - \alpha(\gamma - 1)] + \alpha\gamma\\ &= \lambda^2 + \lambda[1 - \alpha(\gamma - 1)] + \alpha = 0.\end{aligned}$$

The eigenvalues are therefore

$$\lambda = \frac{1}{2}\left\{\alpha(\gamma-1) - 1 \pm \sqrt{[\alpha(\gamma-1)-1]^2 - 4\alpha}\right\}.$$

Suppose that $\alpha(\gamma - 1) \approx 1$. Then we have

$$\lambda \approx \frac{1}{2}\left\{\alpha(\gamma-1) - 1 \pm \sqrt{-4\alpha}\right\} = \frac{1}{2}\left\{\alpha(\gamma-1) - 1 \pm i\sqrt{4\alpha}\right\}.$$

The following conclusions follow:

1. The eigenvalues are a complex-conjugate pair with real part $\alpha(\gamma-1)-1$.

2. By varying either α or γ in this neighborhood of parameter space, the real part of the eigenvalue can cross zero. Thus, we have an Andronov-Hopf bifurcation.

The easiest way to tell what kind of Andronov-Hopf bifurcation we have is by simulation. We have to look at what happens near the bifurcation (but not too near due to very slow dynamics very near the bifurcation). Figure 1 shows the result of such a calculation. It appears that the limit cycles start out small and grow. This suggests that we have a supercritical Andronov-Hopf bifurcation.

A clearer picture emerges if we use AUTO to compute a bifurcation diagram. The bifurcation diagram is shown in Fig. 2. As we have seen previously, the thick red line represents a stable equilibrium point, while the black line represents an unstable equilibrium. The filled green circles in such a diagram are the minimum and maximum values of the variable (here, x) on a stable limit cycle. We can now very clearly see that the limit cycle grows out of the equilibrium point once the latter loses stability. A bifurcation diagram with this appearance is characteristic of a supercritical Andronov-Hopf bifurcation.

The curve in parameter space where $\Re(\lambda) = \alpha(\gamma-1) - 1 = 0$ separates a region where the equilibrium point is stable from one in which it is unstable. This curve is given by

$$\gamma = \frac{1}{\alpha} + 1.$$

Plotting this curve and labeling the different regions with the corresponding behavior produces a **phase diagram**. The phase diagram of the Sel'kov model is shown in Fig. 3. Note the analogy of a phase diagram with phase diagrams from thermodynamics. Also note that a phase diagram has nothing directly to do with phase space.

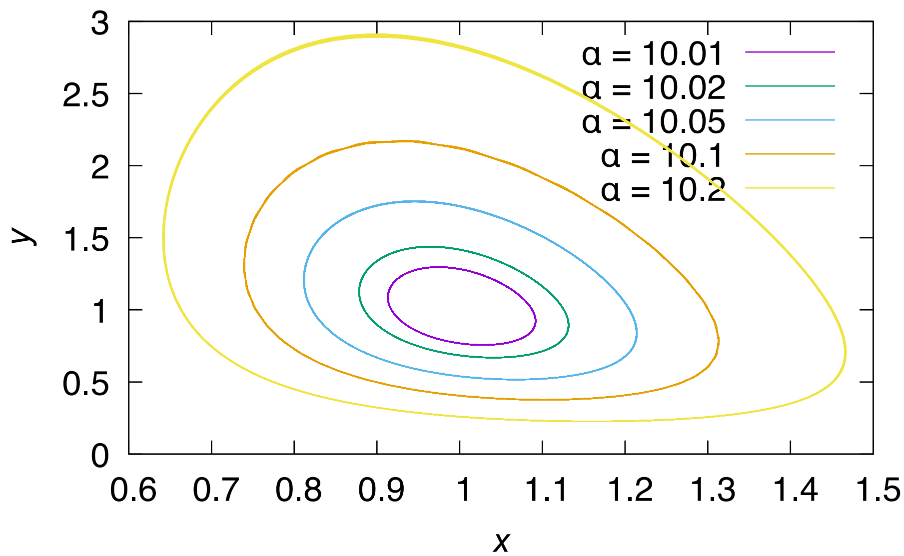

Figure 1: Limit cycles of the Sel'kov model with $\gamma = 1.1$ and various values of α

References

[1] E. E. Sel'kov. Self-oscillations in glycolysis. 1. a simple kinetic model. *Eur. J. Biochem.*, 4:79–86, 1968.

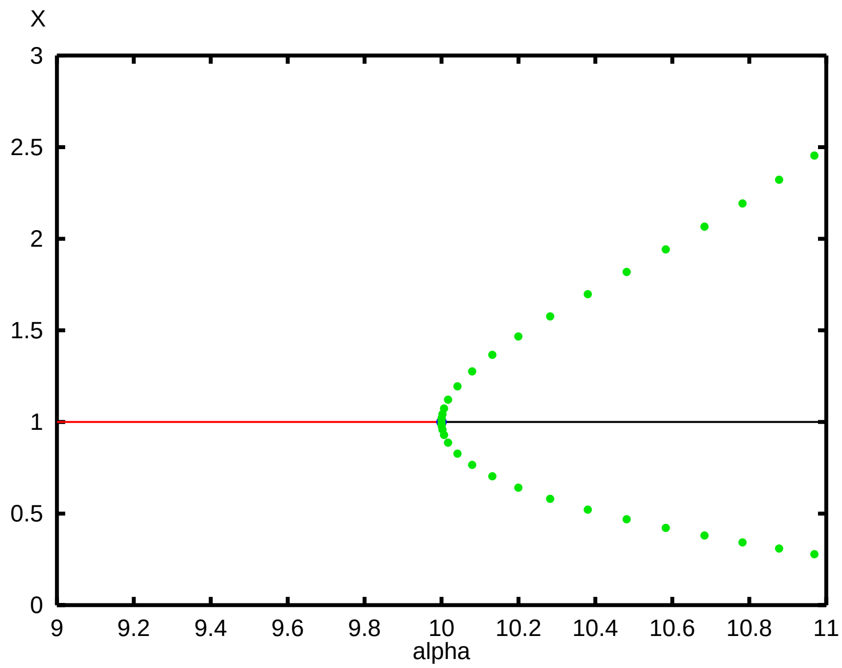

Figure 2: Bifurcation diagram for the Sel'kov model with $\gamma = 1.1$

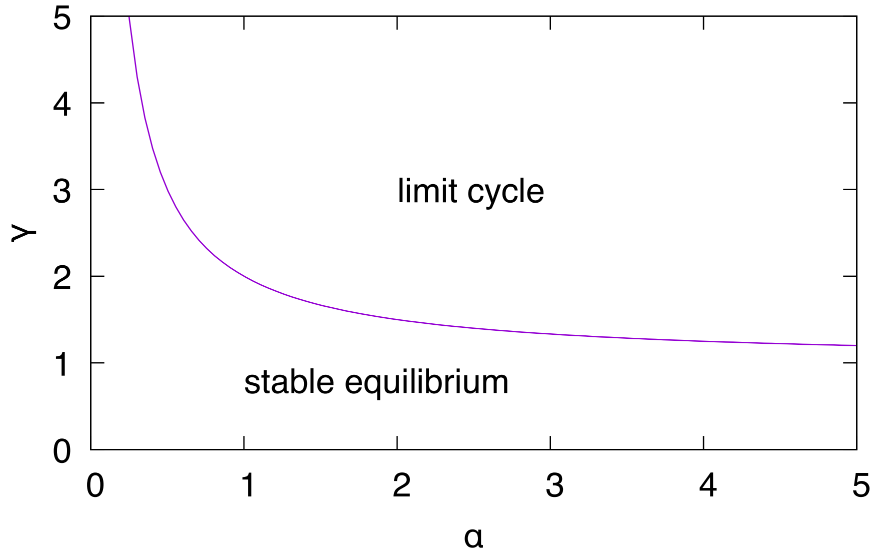

Figure 3: Phase diagram of the Sel'kov model