# Practice exercises for test 1

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September 29, 2019

There is **no assignment** this week so you can focus on studying for the test.

The following problems practice the skills you will need to complete the test, although some of these problems may be slightly harder than test questions.

1. Exercises 2.1–2.3 from the textbook

2. Carry out a linear stability analysis of the equation
$$\dot{x} = \mu + x^2,$$
where the parameter $\mu$ can adopt any value, positive or negative. Identify any bifurcations you encounter.

3. The van der Pol oscillator has a long history in nonlinear dynamics. It was first devised by Balthasar van der Pol while he was working at Philips Research Laboratories, and was eventually used to understand action potentials in electrically excitable cells, especially in the heart. An electrical circuit realizing these equations can be built, and so it can be studied experimentally in a very direct way. There are several different ways of writing the van der Pol equations, including this one:
$$\begin{aligned}\dot{x}_1 &= x_2 + \lambda(x_1 - x_1^3/3),\\ \dot{x}_2 &= -x_1\end{aligned}$$

The variables can be positive or negative. Consider both positive and negative values of $\lambda$.[1]

(a) Carry out a linear stability analysis of the equilibrium point. Does the stability depend on $\lambda$? If so, what type of bifurcation might you expect?

(b) Confirm your analysis using AUTO.

Hint: Start at a value of $\lambda$ where the equilibrium point is stable.

4. The potential energy of many anharmonic oscillators can be approximated by

$$V(x) = \frac{1}{2}k_2x^2 + \frac{1}{4}k_4x^4.$$

(a) The force is related to the potential energy by

$$F = -\frac{dV}{dx}.$$

Starting from $F = ma$, write down a pair of coupled ordinary differential equations for the motion of the oscillator.

Hint: Look at example 3.3 in the textbook.

(b) Is the total energy a Lyapunov function for an anharmonic oscillator with the above potential energy? If so, what does the existence of a Lyapunov function prove?

5. The Oregonator is a classic model for the Belousov-Zhabotinsky (BZ) reaction, arguably the first chemical oscillator to be studied in detail. In the classic BZ reaction, $Ce^{3+}$ is oxidized by $HBrO_3$, and $Ce^{4+}$ is reduced back to cerium(III) by an organic acid, usually malonic acid. There is also some important organobromine chemistry in the mechanism, the details of which are perhaps best left for another time. In its simplest form, the Oregonator takes the form

$$\dot{x} = \mu\left[x(1-x) - fz\frac{x-q}{x+q}\right],$$
$$\dot{z} = x - z.$$

[1] In the physical interpretation of this model, $\lambda$ can only be positive. It is however a useful exercise here to allow $\lambda$ to be negative as well.

The variables $x$ and $z$ represent, respectively, the dimensionless concentrations of $HBrO_2$ and of cerium(IV) ions. The parameters $\mu$, $f$ and $q$ are all positive.

(a) Show that there is a trivial equilibrium point at $(0, 0)$, as well as one positive equilibrium point.

Note: You don't have to solve for the coordinates of the nontrivial equilibrium. I am only asking you to show that such a point must exist.

(b) Using linear stability analysis, show that the trivial equilibrium point is unstable. Classify this equilibrium point as a node, focus or saddle point.

(c) Using the parameters $f = 1$, and $q = 7.6 \times 10^{-5}$, carry out a bifurcation analysis of this mechanism using AUTO. Try starting with a small but nonzero value of $\mu$. Use **nonzero** initial conditions to find an equilibrium point numerically as a starting point for AUTO. By varying $\mu$, you should eventually see an Andronov-Hopf bifurcation. What kind of Andronov-Hopf bifurcation is it?

Hint: After identifying the neighborhood of the Andronov-Hopf bifurcation using AUTO, you will want to take a fairly narrow window of the parameter $\mu$ around this point in order to observe the behavior of the solutions near the bifurcation and thus determine its type.

(d) Why would you not want to start at $\mu = 0$?