Chemistry 4010 Fall 2019 Assignment 1 Solutions

1. (a) Possible units of the variables and parameters:

    P: individuals
    t: s
    r: s^{-1}
    K: individuals

    The following dimensionless variables can be formed:

    $$\tau = rt$$
    $$p = P/K$$

    Starting from the original equation rearrange as follows:

    $$\frac{dP}{r\,dt} = \frac{dP}{d(rt)} = \frac{dP}{d\tau} = P(1 - P/K).$$

    Now divide both sides by K:

    $$\frac{\frac{1}{K}dP}{d\tau} = \frac{d(P/K)}{d\tau} = \frac{dp}{d\tau} = \frac{P}{K}\left(1 - \frac{P}{K}\right) = p(1-p).$$

    The net result is

    $$\dot{p} \equiv \frac{dp}{d\tau} = p(1-p).$$

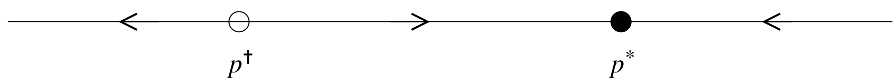
   (b) First, we need to find the equilibrium points: $\dot{p} = 0$ when

    $$p^{\dagger} = 0 \qquad \text{or} \qquad p^{*} = 1.$$

    $\dot{p}$ is a product of two factors, p and $1 - p$. We can determine the sign of $\dot{p}$ by thinking about the signs of these two factors. The line naturally divides into three regions, as follows:

    - If $p < p^{\dagger}$, then $p < 0$ and $1 - p > 0$. Therefore $\dot{p} < 0$.
    - If $p^{\dagger} < p < p^{*}$, then $p > 0$ and $1 - p > 0$. Therefore $\dot{p} > 0$.
    - If $p > p^{*}$, then $p > 0$ and $1 - p < 0$. Therefore $\dot{p} < 0$.

    We can now sketch the flow on the line:



(c) For any $p > 0$, $p \to p^* = 1$. In dimensional form, this is $P/K \to 1$, or $P \to K$. K is called the carrying capacity because it is the size of the population that a given environment can support ("carry").

2. (a) The full set of mass-action differential equations is

$$\begin{aligned}
\frac{d[\mathrm{E}]}{dt} &= -k_1[\mathrm{E}][\mathrm{S}] + k_{-1}[\mathrm{C}] + k_{-2}[\mathrm{C}] \\
\frac{d[\mathrm{S}]}{dt} &= -k_1[\mathrm{E}][\mathrm{S}] + k_{-1}[\mathrm{C}] \\
\frac{d[\mathrm{C}]}{dt} &= k_1[\mathrm{E}][\mathrm{S}] - k_{-1}[\mathrm{C}] - k_{-2}[\mathrm{C}] \\
\frac{d[\mathrm{P}]}{dt} &= k_{-2}[\mathrm{C}]
\end{aligned}$$

[P] doesn't appear in any of the differential equations, so we can neglect the last equation. Note that

$$\frac{d[\mathrm{E}]}{dt} + \frac{d[\mathrm{C}]}{dt} = 0,$$

which implies that $[\mathrm{E}] + [\mathrm{C}] = E_0$, where E_0 is a constant. Thus,

$$[\mathrm{E}] = E_0 - [\mathrm{C}].$$

The rate equations reduce to the planar system

$$\begin{aligned}
\frac{d[\mathrm{S}]}{dt} &= -k_1[\mathrm{S}]\left(E_0 - [\mathrm{C}]\right) + k_{-1}[\mathrm{C}] \\
\frac{d[\mathrm{C}]}{dt} &= k_1[\mathrm{S}]\left(E_0 - [\mathrm{C}]\right) - (k_{-1} + k_{-2})[\mathrm{C}]
\end{aligned}$$

Dimensions of the variables and parameters:

[S], [C]:	M
t:	s
E_0:	M
k_1:	$\mathrm{M}^{-1}\mathrm{s}^{-1}$
k_{-1}, k_{-2}:	s^{-1}

Anticipating later developments in the course,[1] it might make sense to define

$$c = [\mathrm{C}]/E_0,$$

since it is then the case that $0 \leq c \leq 1$. There is not quite so natural a measurement scale for [S]. We choose, arbitrarily,

$$s = k_1[\mathrm{S}]/k_{-1}.$$

Equally arbitrarily,

$$\tau = k_{-2}t.$$

Using these definitions, we get

$$\begin{aligned}
\frac{d[\mathrm{S}]}{dt} &= \frac{k_{-1}k_{-2}}{k_1}\frac{ds}{d\tau} \equiv \frac{k_{-1}k_{-2}}{k_1}\dot{s} \\
\frac{d[\mathrm{C}]}{dt} &= k_{-2}E_0\frac{dc}{d\tau} \equiv k_{-2}E_0\dot{c}
\end{aligned}$$

Therefore

$$\begin{aligned}
\dot{s} &= \frac{k_1}{k_{-1}k_{-2}}\left\{-k_1\frac{k_{-1}}{k_1}s\left(E_0 - E_0c\right) + k_{-1}E_0c\right\} \\
&= -\frac{k_1E_0}{k_{-2}}s(1-c) + \frac{k_1E_0}{k_{-2}}c.
\end{aligned}$$

Define

$$\alpha = k_1E_0/k_{-2}.$$

Therefore

$$\dot{s} = \alpha\left(-s(1-c) + c\right).$$

Similarly,

$$\begin{aligned}
\dot{c} &= \frac{1}{k_{-2}E_0}\left\{k_1\frac{k_{-1}}{k_1}s\left(E_0 - E_0c\right) - (k_{-1} + k_{-2})E_0c\right\} \\
&= \frac{k_{-1}}{k_{-2}}s(1-c) - \left(\frac{k_{-1}}{k_{-2}} + 1\right)c.
\end{aligned}$$

[1]The scaling used here is a completely arbitrary one, intended only to reduce the number of parameters in the model. It is *not* a good scaling for any other purpose.

Define

$$\beta = k_{-1}/k_{-2}.$$

Then

$$\dot{c} = \beta s(1-c) - (1+\beta)c.$$

Our pair of rate equations is therefore

$$\begin{aligned}
\dot{s} &= \alpha\left(-s(1-c)+c\right) \\
\dot{c} &= \beta s(1-c) - (1+\beta)c
\end{aligned}$$

(b) We first need to find the equilibrium point(s):

$$\alpha\left(-s(1-c)+c\right) = 0 \tag{1a}$$

$$\beta s(1-c) - (1+\beta)c = 0 \tag{1b}$$

Solve for s from (1a):

$$s = \frac{c}{1-c}$$

Substitute this expression into (1b):

$$\begin{aligned}
\beta\frac{c}{\cancel{1-c}}\cancel{(1-c)} - (1+\beta)c &= 0 \\
\therefore c^* &= 0 \\
\therefore s^* &= 0
\end{aligned}$$

We will need the nullclines. Typically we would have s (the substrate/reactant concentration) as the abscissa, and c as the ordinate. We will therefore express the nullclines in the form $c(s)$.

$$\begin{aligned}
\dot{s} = 0 \qquad & \Longrightarrow c_{\mathcal{S}} = \frac{s}{s+1} \\
\dot{c} = 0 \qquad & \Longrightarrow c_{\mathcal{C}} = \frac{\beta s}{\beta s + 1 + \beta} = \frac{s}{s+1+1/\beta}
\end{aligned}$$

Since $s+1+1/\beta > s+1$, $c_{\mathcal{C}} < c_{\mathcal{S}}$ at any given value of s. In other words, the s nullcline lies above the c nullcline everywhere. However, as $s \to \infty$, $c_{\mathcal{C}} \to 1$ and $c_{\mathcal{S}} \to 1$, so the two nullclines pinch together at large s. The nullclines are in fact hyperbolas.

Along the s axis, $c = 0$, and the velocity vector becomes $\mathbf{v} = (\dot{s}, \dot{c}) = (-\alpha s, \beta s)$. Imagine starting a trajectory on the s axis.

Here, $\mathbf{v}$ has the sign pattern $(-,+)$. Since the c nullcline is below the s nullcline, we will encounter the c nullcline first. On this nullcline (by definition) $\dot{c} = 0$, so the velocity vector here has the sign pattern $(-,0)$. Note that the signs of the velocity components can only change on a nullcline, which is how we know that $\dot{s}$ is still negative here. After we cross the nullcline, the sign pattern of $\mathbf{v}$ becomes $(-,-)$.

Along the c axis, $s = 0$, and the velocity vector is of the form $\mathbf{v} = (\alpha c, -c(1+\beta))$, i.e. the sign pattern of $\mathbf{v}$ is $(+,-)$. Following a trajectory, we will run into the s nullcline, where $\mathbf{v}$ will have the sign pattern $(0,-)$. After crossing this nullcline, the sign pattern becomes $(-,-)$. This puts us in the region between the nullclines. Note that the same sign pattern has to be obtained in this region whether you approach from the top or bottom, i.e. by crossing one nullcline or the other.

We can now sketch the vector field, shown in Fig. 1.

We can now sketch the flow, essentially by tracing out curves that follow the arrows. My sketch is shown in Fig. 2.

(c) My `Xppaut` input file is the following:

```
# Assignment 1, Michaelis-Menten mechanism
ds/dt = alpha*(-s*(1-c)+c)
dc/dt = beta*s*(1-c)-(1+beta)*c

param alpha=1, beta=1

done
```

The result of the "flow" and "nullcline" commands is shown in Fig. 3. Your results may differ slightly depending on the parameters you chose, on the scaling of your axes, and on the number of grid points you used.

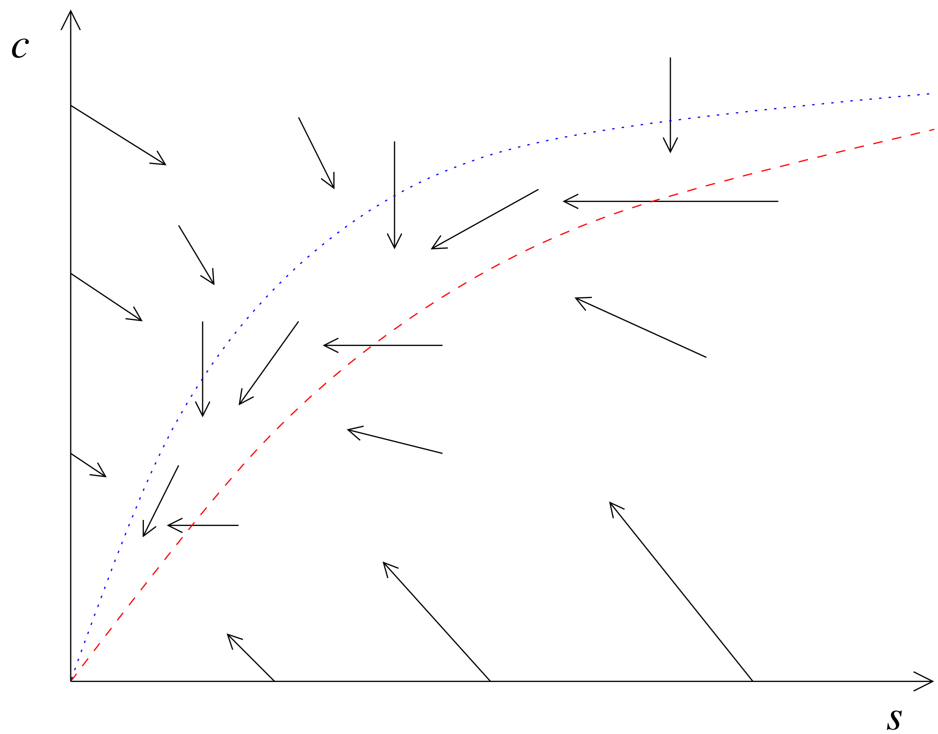


Figure 1: Vector field for the Michaelis-Menten mechanism. The dashed red curve is the c nullcline. The dotted blue curve is the s nullcline.

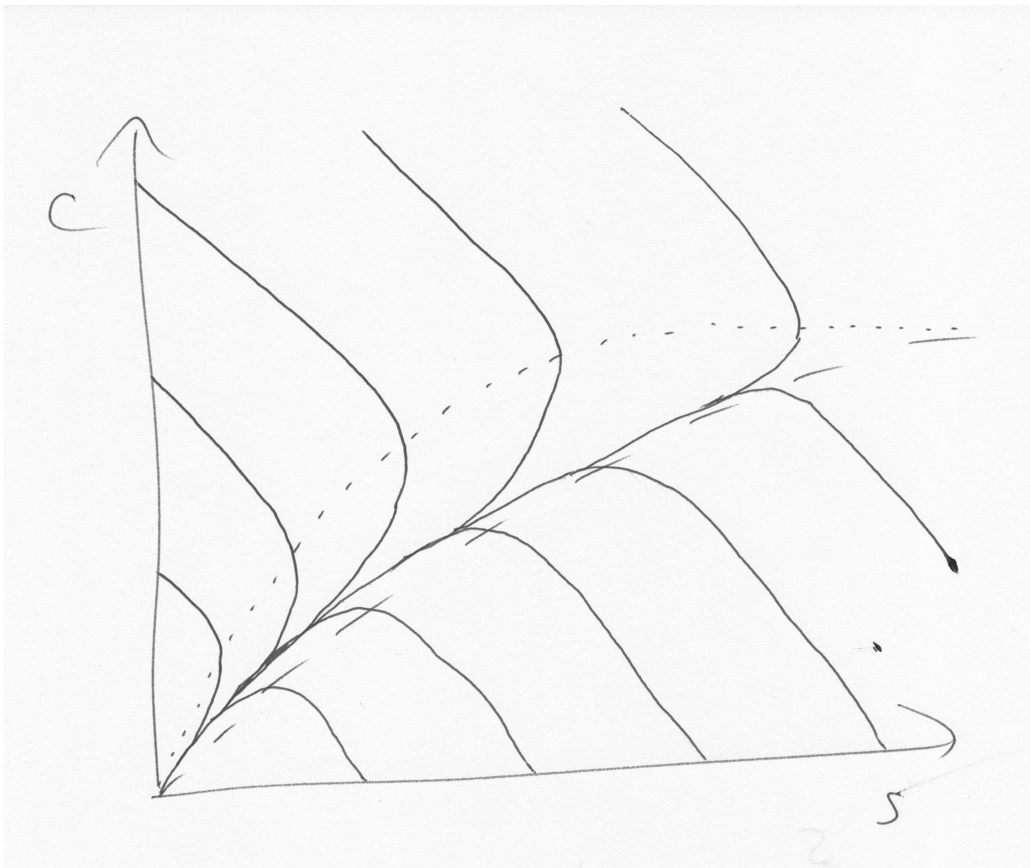


Figure 2: Sketch of the trajectories (flow) for the Michaelis-Menten mechanism. The dashed curve is the c nullcline. The dotted curve is the s nullcline.

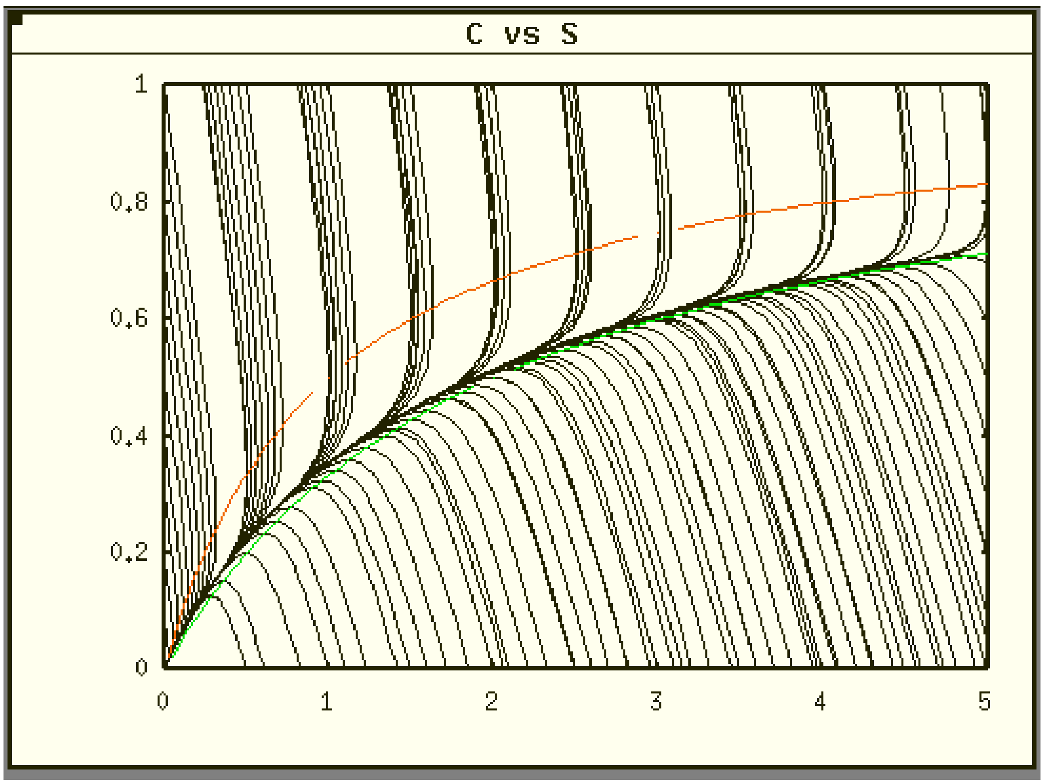


Figure 3: Michaelis-Menten flow and nullclines computed by XPPAUT.