The Margulis Normal Subgroups Theorem
Lecture 2: Statement and Most of the Proof

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Introduction

To minimize the amount of Lie theory needed, we will prove only a special case of the theorem:

$\mathrm{SL}(3,\mathbf{Z})$ *is "simple modulo finite groups."*

Recall. Γ simple:

$N \triangleleft \Gamma \Rightarrow N$ trivial or $N = \Gamma$ (i.e., Γ/N trivial)

Theorem. *Let* $\Gamma = \mathrm{SL}(3,\mathbf{Z})$.

$N \triangleleft \Gamma \Rightarrow N$ *is finite or* Γ/N *is finite.*

Remark.

- **False** for $\Gamma = \mathrm{SL}(2,\mathbf{Z}) \approx$ free group
  (free groups have *many* normal subgroups)
- **False** for Gromov hyperbolic groups;
  e.g., lattices in:
  - $\mathrm{SO}(1,n)$ real ($\mathbf{R}$) hyperbolic
  - $\mathrm{SU}(1,n)$ complex ($\mathbf{C}$) hyperbolic
  - $\mathrm{Sp}(1,n)$ quaternion ($\mathbf{H}$) hyperbolic
  - $F_{4,1}$ octonion ($\mathbf{O}$) hyperbolic
- The proof of Margulis works for lattices in any other noncompact simple Lie group (such as $\mathrm{SU}(2,n)$, with $n \geq 2$).
- Many cases (including $\mathrm{SL}(3,\mathbf{Z})$) were known before the work of Margulis (1978).

The proof has 3 steps:

1. Assume N is infinite. Then Γ/N is amenable.
2. Γ has Kazhdan's property (T), so Γ/N has (T).
   (We will assume this without proof.)
3. (easy) amenable + T $\Rightarrow$ finite (for discrete groups)

Proof of Step 1 (Γ/N is amenable).

Suppose Γ/N acts on compact, metrizable X.

Want: $\exists$ invariant probability measure on X.

Key Lemma. $\exists$ (*essentially*) Γ*-equivariant measurable* $\psi\colon G/P \to \mathrm{Prob}(X)$.

Black Box. $\psi\colon G/P \to Z$ (Γ*-equivariant*)

$\Longrightarrow$ *action of* Γ *on* Z *extends to action of* G (*a.e.*) *s.t.* ψ *is (essentially)* G*-equivariant.*

N acts trivially on X, so it acts trivially on $\mathrm{Prob}(X)$.
Thus, the kernel of the G-action is infinite.
G is simple (mod finite center),
so this implies the kernel is all of G.
I.e., the action of G on $\mathrm{Prob}(X)$ is trivial.
So the action of Γ on $\mathrm{Prob}(X)$ is (essentially) trivial.
Therefore X has an invariant measure.

Proof of Step 3 (amenable + T $\Rightarrow$ finite).

Definition. G has Kazhdan's property (T):

$\forall \rho\colon G \to \mathcal{U}(\mathcal{H})$,

$\exists$ almost-invariant vectors $\Rightarrow \exists$ invariant vectors.

Theorem (Kazhdan, 1967).
$\mathrm{SL}(3,\mathbf{R})$ *and* $\mathrm{SL}(3,\mathbf{Z})$ *have* (T).

Proposition. G *has* (T), $N \triangleleft G \Rightarrow G/N$ *has* (T).

Proposition.
G *amenable and has* $(T) \Rightarrow G$ *is compact.*

Proof. G amen $\Rightarrow \mathcal{L}^2(G)$ has almost-invariant vectors

$\Rightarrow \mathcal{L}^2(G)$ has an invariant vector

$\Rightarrow 1_G \in \mathcal{L}^2(G)$

$\Rightarrow \mu(G) < \infty$

$\Rightarrow G$ is compact. □

References

D. A. Kazhdan:
Connection of the dual space of a group with the structure of its closed subgroups,
Func. Anal. Appl. 1 (1967) 63–65.
MR0209390 (35 #288)
(The original paper on Kazhdan's Property (T))

G. A. Margulis:
Discrete Subgroups of Semisimple Lie Groups.
Springer, Berlin Heidelberg New York, 1991.
ISBN: 3-540-12179-X; MR1090825 (92h:22021)
(Section 4.4 proves the Normal Subgroups Theorem; see Chapter 3 for Kazhdan's Property (T))

G. A. Margulis:
Quotient groups of discrete subgroups and measure theory,
Func. Anal. Appl. 12 (1978), no. 4, 295–305 (1979)
MR0515630 (80k:22005)
(The original publication of this proof.)

Robert J. Zimmer:
Ergodic Theory and Semisimple Groups.
Birkhäuser, Basel, 1984.
ISBN: 3-7643-3184-4; MR0776417 (86j:22014)
(Chapter 8 proves the Normal Subgroups Theorem; Section 7.4 proves (T) For $\mathrm{SL}(3,\mathbf{R})$ and $\mathrm{SL}(3,\mathbf{Z})$)