Automorphisms of direct products of some circulant graphs

Dave Witte Morris
University of Lethbridge, Alberta, Canada
https://deductivepress.ca/dmorris
dmorris@deductivepress.ca

Abstract. The direct product of two graphs X and Y is denoted $X \times Y$. Its automorphism group contains a copy of the direct product of $\mathrm{Aut}(X)$ and $\mathrm{Aut}(Y)$, but it is not known when this inclusion is an equality, even for the special case where X is a circulant graph and $Y = K_2$ is a connected graph with only 2 vertices. Joint work with Ademir Hujdurović and Đorđe Mitrović sheds some light on this special case, including a complete answer when the valency of X is less than 8.

https://deductivepress.ca/dmorris/talks/AutDirProd-Waterloo2022.pdf

Graph products

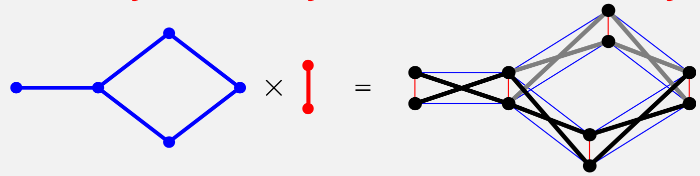
Given two graphs X and Y, construct a new graph $X * Y$.
Most important: Cartesian $\square$, strong $\boxtimes$, direct $\times$.

$$V(X * Y) = \{ (x, y) \mid x \in V(X),\ y \in V(Y) \} = V(X) \times V(Y)$$

Definition (direct product $\times$)

(categorical product, tensor product, Kronecker product, ...)

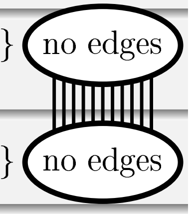
$$(x_1, y_1) \overset{X \times Y}{-\!\!-} (x_2, y_2) \iff x_1 \overset{X}{-\!\!-} x_2 \text{ and } y_1 \overset{Y}{-\!\!-} y_2$$

Eg. 

- commutative: $X * Y \cong Y * X$
- associative: $(X * Y) * Z \cong X * (Y * Z)$

Note. $X \times K_2$ is **bipartite**.
Canonical bipartite double cover of X.

$V(X) \times \{1\}$
$V(X) \times \{0\}$



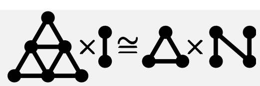
Exercise

Choose a graph product $*$ $(= \square, \boxtimes, \times)$. $(V(X * Y) = V(X) \times V(Y))$
Show that every (finite) graph X has a prime decomposition for $*$:

- $X \cong X_1 * X_2 * \cdots * X_n$.
- No X_i can be written as $Y * Z$ (with Y, Z smaller than X_i).

Theorem (Sabidussi-Vizing 1960/1963, Dörfler-Imrich 1970)

X connected $\Rightarrow$ prime decomposition is unique for $\square$ and $\boxtimes$.
(up to permutation of the factors and isomorphism)

Fact. Prime decomposition is **not** unique for $\times$: 

Rem. Prime decomp is not unique for $\square$ if graphs not connected:
$(1{+}x{+}x^2)(1{+}x^3) = (1{+}x{+}\cdots{+}x^6) = (1{+}x^2{+}x^4)(1{+}x)$ in $\mathbb{Z}^+[x]$
is a non-unique prime factorization.
Let x = graph $(= K_2)$. (+ is disjoint union and $x^n = x \square x \square \cdots \square x$)

$\square, \boxtimes, \times$ are natural graph-theoretic constructions:

$$X \overset{\alpha}{\cong} X',\ Y \overset{\beta}{\cong} Y' \ \Rightarrow\ X * Y \overset{\alpha \times \beta}{\cong} X' * Y'.$$

So $\mathrm{Aut}\, X \times \mathrm{Aut}\, Y \subseteq \mathrm{Aut}(X * Y)$.

Exercise

$\mathrm{Aut}\, X \times \mathrm{Aut}\, Y = \mathrm{Aut}(X * Y) \Rightarrow X$ relatively prime to Y for $*$.

Theorem (Sabidussi-Vizing 1960/1963)

Converse is true for $\square$. (if X and Y are connected)

Also for $\boxtimes$, but need an additional technical condition.

Bad news

Converse is **not** true for $\times$:
we do not understand $\mathrm{Aut}(X \times Y)$, even if $Y = K_2 =$ ●—●.

Defn (Marušič-Scapellato -Zagaglia 1989). X is ***unstable*** if $\mathrm{Aut}(X \times K_2) \neq \mathrm{Aut}\, X \times \mathrm{Aut}\, K_2$.

Defn (Marušič et al.). X is ***unstable*** if $\operatorname{Aut}(X \times K_2) \neq \operatorname{Aut} X \times \operatorname{Aut} K_2$.

Definition (Lauri -Mizzi-Scapellato 2019)

$(\alpha_0, \alpha_1) \in S_{V(X)} \times S_{V(X)}$ is a ***2-fold automorphism*** if

$v \text{ — } w \iff \alpha_0(v) \text{ — } \alpha_1(w)$

Obvious: $\alpha_0 = \alpha_1 \in \operatorname{Aut}(X)$.

Exercise (Marušič-Scapellato-Zagaglia 1989)

Assume $X \times K_2$ is connected (i.e., X is connected and not bipartite).
Show: X is unstable $\iff$ X has a nonobvious 2-fold automorphism.

Hint: ($\Leftarrow$) Define $\alpha(x, i) = (\alpha_i(x), i)$.

Exercise (an obvious cause of instability)

X is unstable if X has "twin" vertices. even if connected

Hint: Assume $\{$neighbours of $a\} = \{$neighbours of $b\}$. ("twins")
$\exists\ \alpha_0 \in \operatorname{Aut} X$ that interchanges a and b, but fixes all other verts.
Let $\alpha_1(v) = v$.

Bad news

We do not understand $\operatorname{Aut}(X \times Y)$, even if $Y = K_2 = \bullet\!\!-\!\!\bullet$.

Good news

The problem only arises for graphs that are *bipartite.*

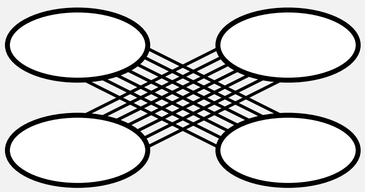
Theorem (Dörfler 1974)

$\operatorname{Aut}(X \times Y) = \operatorname{Aut} X \times \operatorname{Aut} Y$ *if X and Y are connected, twin-free, and **not** bipartite and X is $\times$-coprime to Y.*

Exercise

Assume X and Y are bipartite
(and have more than one vertex).

1. Show $X \times Y$ is not connected.
2. Show $\operatorname{Aut}(X \times Y) \neq \operatorname{Aut} X \times \operatorname{Aut} Y$ if $\operatorname{Aut} X$ and $\operatorname{Aut} Y$ are nontrivial.



Bad news: We do not understand $\operatorname{Aut}(X \times Y)$, even if $Y = K_2 = \bullet\!\!-\!\!\bullet$.

- **both** X and Y not bipartite: **good** [Dörfler]
- **both** X and Y bipartite: **bad** [exercise]

Open case: X is not bipartite and Y is bipartite.

The simplest nontrivial bipartite graph is K_2.
That is one reason why it is important to study $\operatorname{Aut}(X \times K_2)$.
(Another reason: $X \times K_2$ is the canonical double cover.)
But it is not just a special case — it is the main case:

Proposition (classical?)

Assume $\operatorname{Aut}(X \times K_2) = \operatorname{Aut} X \times \operatorname{Aut} K_2$. *(and X is not bipartite)*
Then $\operatorname{Aut}(X \times Y) = \operatorname{Aut} X \times \operatorname{Aut} Y$
if X is coprime to Y in an appropriate sense.

Eg., If X and Y are circulant graphs, then suffices to assume
$\gcd(\#V(X), \#V(Y)) = 1$.

Defn (Marušič et al.). X is ***unstable*** if $\operatorname{Aut}(X \times K_2) \neq \operatorname{Aut} X \times \operatorname{Aut} K_2$.

Open problem (Steve Wilson 2008)

Which connected, nonbipartite, twin-free circulant graphs are unstable?

Theorem (Fernandez-Hujdurović 2022)

None *if $\#$ vertices is odd.*

Theorem (Hujdurović-Mitrović-Morris 2021)

None *of order n*
$\iff$ *n is odd or $n \leq 8$ or $n = 2p$ where p is prime and $p \equiv 3 \pmod 4$.*

Open problem (Steve Wilson 2008)

Which connected, twin-free, nonbipartite, circulant graphs are unstable?

Assume X is a connected, twin-free, nonbipartite, circulant graph.
Also assume $|V(X)|$ is even.
Characterize cases where $\operatorname{Aut}(X \times K_2) \neq \operatorname{Aut} X \times \operatorname{Aut} K_2$.

Uncommon (?): for order ≤ 50, $\approx 70{,}000/3{,}600{,}000 < 2\%$ (?)

Partial answer [Wilson]: 4 conditions (C.1 – C.4) that imply unstable.

Proposition (Wilson type C.4)

$\operatorname{Circ}(n; S)$ *is unstable if* $\exists m \in \mathbb{Z}_n^\times$, *such that* $mS = S + \frac{n}{2}$.

Proof. Let $\alpha_0(x) = mx$ and $\alpha_1(x) = \alpha_0(x) + \frac{n}{2}$. □

Easy generalization (Hujdurović-Mitrović-Morris 2021)

$$\text{Unstable if } \operatorname{Circ}(n; S) \overset{\alpha_0}{\cong} \operatorname{Circ}\left(n;\ S + \frac{n}{2}\right).$$

Open problem (Steve Wilson 2008)

Which connected, twin-free, nonbipartite, circulant graphs are unstable?

Partial answer [Wilson]: 4 conditions (C.1 – C.4) that imply unstable.

Here is another one: ($S_e = S \cap 2\mathbb{Z}_n$ and $S_o = S \setminus S_e$)

Proposition (Wilson type C.1)

$\operatorname{Circ}(n; S)$ *is unstable if* $\exists$ *nontrivial subgroup* H *of* $\mathbb{Z}_n$, $S_e + H = S_e$.

Proof. Fix $h \in H \setminus \{0\}$. Let $\alpha_i(v) = \begin{cases} v & \text{if } v \equiv i \pmod 2; \\ v + h & \text{if } v \not\equiv i \pmod 2. \end{cases}$ □

Can generalize to also include (C.2) and (C.3) as special cases.

Proposition (Hujdurović-Mitrović-Morris 2021)

Let H, K *nontriv subgrps of* $\mathbb{Z}_n$ *with* $|K|$ *even.* $\operatorname{Circ}(n; S)$ *unstable if*

1. $S + H \subseteq S \cup (K_o + H)$ *and* $H \cap K_o = \emptyset$, *or*
2. $(S \setminus K_o) + H \subseteq S \cup K_o$ *and either* $|H| \neq 2$ *or* $|K|$ *is divisible by* 4.

(C.1 is the special case with $K = \mathbb{Z}_n$)

Open problem (Steve Wilson 2008)

Which connected, twin-free, nonbipartite, circulant graphs are unstable?

Wilson [2008] conjectured that (C.1 – C.4) are a complete answer.
Counterexample: $\operatorname{Circ}(24; \pm 2, \pm 3 \pm 8, \pm 9, \pm 10)$ [Qin-Xia-Zhou].
Our generalizations provide infinite families of counterexamples.

On the other hand, by lengthy case-by-case analysis:

Theorem (Hujdurović-Mitrović-Morris 2022⁺)

Wilson found all unstable circulants of valency ≤ 7.

Counterexample of valency 8: $\operatorname{Circ}(48; \pm 3, \pm 4, \pm 6, \pm 21)$.

In fact, we list all unstable circulants with valency ≤ 7.

Example (Hujdurović-Mitrović-Morris 2022⁺)

Unstable connected, nonbipartite, twin-free circulants of valency 5 are:

- $\operatorname{Circ}(12k; \pm s, \pm 2k, 6k)$ with s odd, which has Wilson type C.1.
- $\operatorname{Circ}(8; \pm 1, \pm 3, 4)$, which has Wilson type C.3.

Open problem (Steve Wilson 2008)

Which connected, twin-free, nonbipartite, circulant graphs are unstable?

Partial answer: our generalizations of Wilson (C.1 – C.4).

Another infinite family:
The connected components of $\operatorname{Circ}(n; S_e)$ are unstable,
and S_o is invariant under sufficiently many translations.

Proposition (Hujdurović-Mitrović-Morris 2021)

$\operatorname{Circ}(n; S)$ *is unstable if*

- $\operatorname{Circ}\left(\frac{n}{2}; \frac{1}{2} S_e\right)$ *has a nonobvious 2-fold automorphism* (α_0, α_1),
- $H = \langle\, \alpha_0(v) - v,\ \alpha_1(v) - v \mid v \in \mathbb{Z}_{n/2} \,\rangle$, *and*
- $S_o + 2H = S_o$.

Proof.

$\hat{\alpha}_i(v) = v$ for odd v. $\hat{\alpha}_i(2v) = 2\alpha_i(v)$. □

Defn (Marušič et al. 1989). X is ***unstable*** if $\mathrm{Aut}(X \times K_2) \neq \mathrm{Aut}\, X \times \mathrm{Aut}\, K_2$.

Open problem (Steve Wilson 2008)

Which circulant graphs are unstable?

Assume connected, twin-free, nonbipartite.

"nontrivially unstable"

We generalized Wilson's families of examples (C.1 – C.4) and added another infinite family.

According to a computer search, these families include all of the nontrivially unstable circulant graphs with no more than 50 vertices.

But there may be ∞ more examples yet to be found.

We still do not understand $\mathrm{Aut}(X \times Y)$,
even if X *is circulant and* $Y = K_2$.

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