Maps: Stability and bifurcation analysis

Marc R. Roussel

November 15, 2005

In our previous set of notes, we examined the connections between differential equations and maps. Maps also arise directly in certain applications, so we have good reason to understand their behavior. There is a sophisticated theory of bifurcations in maps which we will, again, only consider in a very cursory way. For simplicity in this document, we will consider only one-dimensional maps. However, all the ideas presented here can be extended to higher-dimensional maps.

1 Linear stability analysis of fixed points

Suppose that we are studying a map

$$x_{n+1} = f(x_n). \tag{1}$$

A **fixed point** is a point for which $x_{n+1} = x_n = x^* = f(x^*)$, i.e. a fixed point is an equilibrium point of a map.

As with differential equations, the study of the stability of fixed points of maps generally proceeds by considering a small displacement from the fixed point. Accordingly, let $\delta x_n = x_n - x^*$. Then, taking a Taylor expansion of the map around x^*, we get

$$x^* + \delta x_{n+1} = f(x^* + \delta x_n) = f(x^*) + f'(x^*)\delta x_n + \ldots,$$

where $f' = df/dx$. Since $x^* = f(x^*)$, this simplifies to

$$\delta x_{n+1} = f'(x^*)\delta x_n + \ldots.$$

For sufficiently small δx_n, we can truncate the Taylor expansion at this term.

There are, as with differential equations, three cases:

1. If $|f'(x^*)| < 1$, then $|\delta x_{n+1}|$ is smaller than $|\delta x_n|$, i.e. the displacement from the fixed point shrinks as we iterate the map. The fixed point is therefore **stable**. Note that if $f'(x^*)$ is negative, the sign of δx_n alternates from iterate to iterate, but this does not alter the overall conclusion.

2. If $|f'(x^*)| > 1$, then the displacement from the fixed point grows with iteration, and the fixed point is therefore **unstable**.

3. If $|f'(x^*)| = 1$ then, to linear order, we can't decide on the stability.

Example 1.1 Consider the map

$$x_{n+1} = f(x_n) = ax_n^2.$$

This map clearly has two fixed points, at $x^\dagger = 0$ and at $x^* = 1/a$. The stability analysis of these two fixed points is straightforward:

$$\begin{aligned} f'(x) &= 2ax. \\ \therefore f'(x^\dagger) &= 0. \\ \text{Also,} \qquad f'(x^*) &= 2. \end{aligned}$$

According to our criteria for stability, the first of these fixed points is always *locally* stable, while the other is always unstable.

If we pick a sufficiently large x_0, then the iterates clearly diverge. It is only for points which are reasonably close to 0 that $x^\dagger$ is an attractor. For this simple map with real initial data, the iterates will converge to 0 if

$$|x_1| = |a|x_0^2 < |x_0|.$$

Rearranging the inequality, we get

$$|x_0| < \frac{1}{|a|}.$$

This inequality defines the **basin of attraction** of the fixed point $x^\dagger$.

Example 1.2 Consider the logistic map

$$x_{n+1} = f(x_n) = \lambda x_n(1 - x_n)$$

with initial data $0 \leq x_0 \leq 1$. In the applications where this map arises, λ is generally a positive parameter.

The fixed points are solutions of

$$x^* = \lambda x^*(1 - x^*).$$

These solutions are

$$\begin{aligned} x^\dagger &= 0, \\ \text{or} \quad x^* &= 1 - \frac{1}{\lambda}. \end{aligned}$$

In order to determine the stability of these points, we need

$$f'(x) = \lambda(1 - 2x).$$

For the first fixed point, we have

$$|f'(x^\dagger)| = \lambda.$$

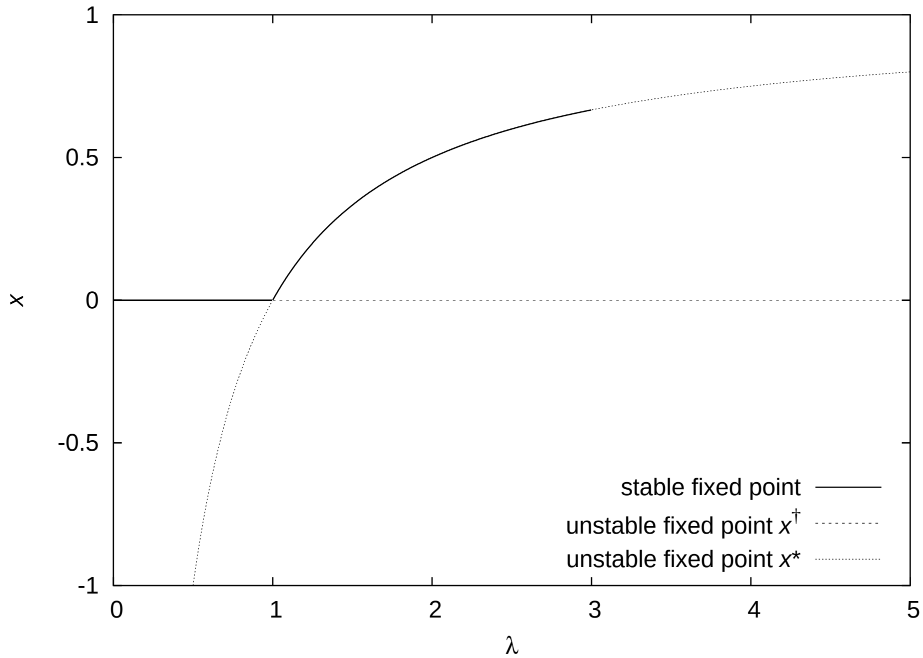


Figure 1: Regions of stability of the two fixed points of the logistic map as a function of λ.

Since a fixed point is stable if $f'(x^\dagger) < 1$, this fixed point is stable if $\lambda < 1$.

For the second fixed point,

$$|f'(x^*)| = \left|\lambda\left[1-2\left(1-\frac{1}{\lambda}\right)\right]\right| = |2-\lambda|\,.$$

If we now apply the stability criterion $|f'(x^*)| < 1$, we get a pair of conditions for stability:

1. $f'(x^*) < 1 \Rightarrow 2-\lambda < 1 \Rightarrow \lambda > 1$.
2. $-f'(x^*) < 1 \Rightarrow \lambda - 2 < 1 \Rightarrow \lambda < 3$.

We conclude that this fixed point is stable in the range $1 < \lambda < 3$.

Note that at $\lambda = 1$, $x^* = x^\dagger = 0$ and that the two equilibrium points exchange stability at this value of λ. A collision of equilibrium points as we vary a parameter with exchange of stability is called a **transcritical bifurcation**. The transcritical bifurcation is one of the most common types of bifurcations in dynamical systems.

Given the information we have collected, we can draw a portion of the bifurcation diagram of the logistic map, shown in Fig. 1.

2 Stability of periodic orbits

Maps also give rise to periodic orbits. A periodic orbit of the map 1 is characterized as follows:

Definition 1 *A **period-**k **point** x^* is a point for which $x_{n+k} = x_n$.*

Since $x_{n+1} = f(x_n)$ and $f_{n+2} = f(x_{n+1}) = f(f(x_n))$, and so on, $x_{n+k} = f^{(k)}(x_n)$, where $f^{(k)}$ is the k-fold composition of the map, i.e. the result of applying the map to itself k times. The k-fold composition is also called the **k'th power** of the map. It follows that a period-k point obeys $x^* = f^{(k)}(x^*)$.

If you think about it for a minute, you will realize that any fixed point of the map is also a period-k orbit since it repeats (trivially) every k steps. Similarly, a period-2 orbit is also an orbit of period 2^n, a period-3 orbit is also a period-3^n orbit, and so on. When we find period-k orbits, we will therefore typically also find a number of lower-period orbits. Once we have discarded these lower-period orbits, we should be left with k points, all of which belong to the same orbit. On rare occasion, we may find several (true) period-k orbits, in which case we would get nk period-k points, belonging to n different orbits.

Note that the composition $x_{n+k} = f^{(k)}(x_n)$ is itself a map. Period-k points are therefore fixed points of this map, and we can apply the same techniques to study the stability of periodic orbits as we did to study the stability of fixed points. Also note that it is sufficient to study the stability of *one* of the period-k points belonging to a given orbit since the entire orbit will necessarily share the same stability properties.

Example 2.1 Let us pursue our analysis of the logistic map. Period-2 points are found by computing fixed points of

$$\begin{aligned} f^{(2)}(x) &= f(f(x)) \\ &= \lambda^2 x(1-x)\left[1-\lambda x(1-x)\right]. \end{aligned}$$

The fixed points satisfy

$$\begin{aligned} x &= \lambda^2 x(1-x)\left[1-\lambda x(1-x)\right], \\ \text{or} \qquad & x\left\{\lambda^2(1-x)\left[1-\lambda x(1-x)\right]-1\right\} = 0. \end{aligned}$$

$x = 0$ is clearly a fixed point of this equation. This is the expected appearance of the fixed points of the map itself among the period-2 orbits. The other fixed point (which we called x^* in example 1.2) should also be a period-2 point. If we use Maple to factor the term in braces, we get

```
> factor(lambda^2*(1-x)*(1-lambda*x*(1-x))-1);
```

$$-(\lambda x+1-\lambda)\,(\lambda^2 x^2-\lambda^2 x-\lambda x+\lambda+1)$$

The first factor gives us the period-1 fixed point we already knew about. The real period-2 orbit is therefore found by solving

$$\lambda^2 x^2 - \lambda x\,(\lambda+1)+\lambda+1.$$

Our two period-2 points are therefore

$$\begin{aligned} x^\ddagger &= \frac{1}{2\lambda}\left\{\lambda+1\pm\sqrt{(\lambda+1)^2-4(\lambda+1)}\right\} \\ &= \frac{1}{2\lambda}\left\{\lambda+1\pm\sqrt{\lambda^2-2\lambda-3}\right\} \\ &= \frac{1}{2\lambda}\left\{\lambda+1\pm\sqrt{(\lambda-3)(\lambda+1)}\right\}. \end{aligned}$$

Following the discussion above, these points points are part of the same orbit. We therefore only need to study the stability of one of these points to determine the stability of the orbit.

Let us now carry out the stability analysis. Again, Maple is a big help here:

```
> f2 := lambda^2*x*(1-x)*(1-lambda*x*(1-x));
```

$$f2 := \lambda^2 x\,(1-x)\,(1-\lambda x\,(1-x))$$

```
> diff(f2,x);
```

$$\lambda^2\,(1-x)\,(1-\lambda x\,(1-x))-\lambda^2 x\,(1-\lambda x\,(1-x))+\lambda^2 x\,(1-x)\,(-\lambda\,(1-x)+\lambda x)$$

```
> df2dx := factor(%);
```

$$df2dx := -\lambda^2\,(-1+2x)\,(2\lambda x^2-2\lambda x+1)$$

```
> xddag := (lambda+1+sqrt((lambda-3)*(lambda+1)))/(2*lambda);
```

$$xddag := \frac{\lambda+1+\sqrt{(\lambda-3)\,(\lambda+1)}}{2\lambda}$$

```
> subs(x=xddag,df2dx);
```

$$-\lambda^2\,(-1+\frac{\lambda+1+\sqrt{(\lambda-3)\,(\lambda+1)}}{\lambda})\,(\frac{(\lambda+1+\sqrt{(\lambda-3)\,(\lambda+1)})^2}{2\lambda}-\lambda-\sqrt{(\lambda-3)\,(\lambda+1)})$$

```
> simplify(%);
```

$$4-\lambda^2+2\lambda$$

The period-2 point is stable if this last value, which is $f^{(2)}(x^\ddagger)$, is less than 1 in absolute value. There are two cases:

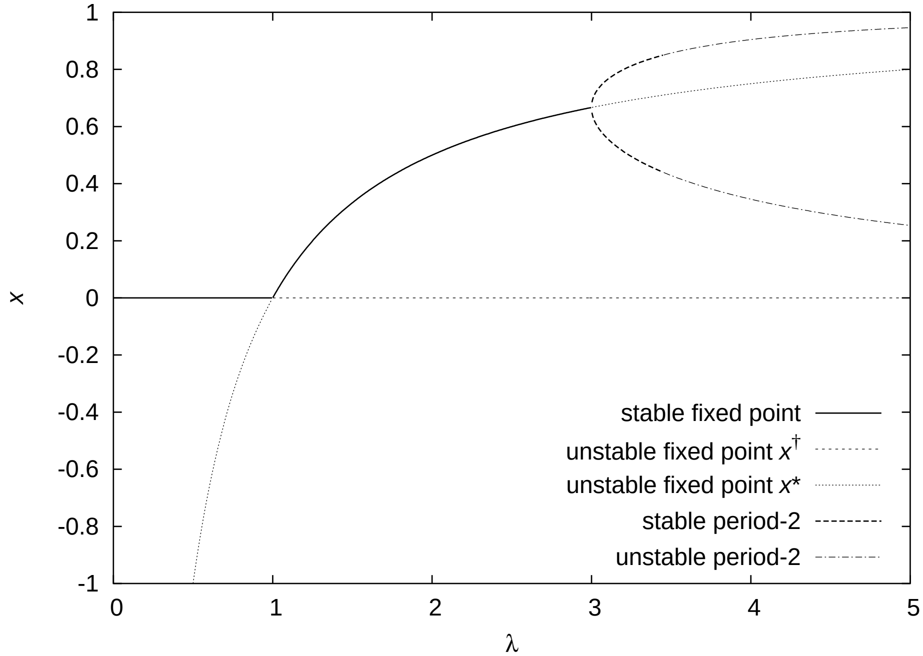


Figure 2: Regions of stability of the period-1 and -2 orbits of the logistic map as a function of λ.

1.
$$\begin{aligned} 4-\lambda^2+2\lambda &< 1. \\ \therefore \lambda^2-2\lambda-3 &> 0. \\ \therefore (\lambda-3)(\lambda+1) &> 0. \\ \therefore \lambda &> 3. \end{aligned}$$

   This last inequality holds because we are restricting our attention to positive values of λ.

2.
$$\begin{aligned} -(4-\lambda^2+2\lambda) &< 1. \\ \therefore \lambda^2-2\lambda-5 &< 0. \\ \therefore \lambda &< 1+\sqrt{6}. \end{aligned}$$

   To get this last result, find the zeros of the polynomial, then note that λ can't be too big to satisfy the middle inequality.

Putting these two results together, we find that the period-2 orbit is stable for $3 < \lambda < 1+\sqrt{6}$. We can add another piece to our bifurcation diagram, shown in Fig. 2.

Finding periodic orbits and determining their stability gets harder and harder for higher periods, even with the help of Maple. Fortunately, `xpp` can do some of this work for us. We start with the following, very simply input file:

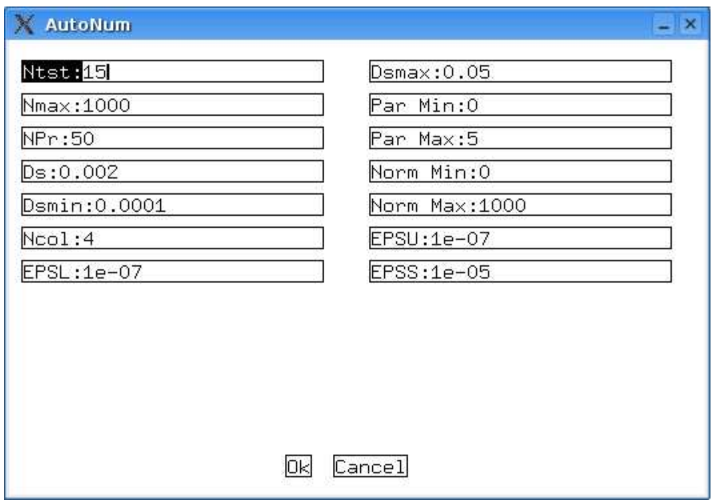
```
# Standard version of the logistic map
x(t+1) = lambda*x*(1-x)

x(0)=0

param lambda=0

done
```

Note that we have set the initial condition and λ at values which correspond to a stable steady state. Suppose that we wanted to locate all the orbits with periods 1, 2, 4 and 8. Click on `nUmerics`→`nOutput` and set this parameter to 8. What this does is that it only passes every eighth iterate to AUTO, so that AUTO is effectively studying the map $f^{(8)}$. Since we have already set the initial conditions to a steady state, we can then go straight to AUTO. Set up the `Axes` to display λ from 0 to 5 and x from 0 to 1.1. Set the `Numerics` as follows:



When you then click on `Run`, you will get the bifurcation diagram displayed in Fig. 3. The bifurcation points can either be read off from the terminal output, or you can use the `Grab` function to go find these points. In the terminal output, these points are denoted either as BP's (branch points) or as HB's (Hopf bifurcations). Note also that some of the branches are missing, but that this is still a pretty good way to get a quick survey of the bifurcation diagram.

Before we leave this topic, note that not every periodic orbit has a period of 2^n. For instance, the logistic map has a stable period-3 orbit in a narrow range of values of λ.

3 Liapunov exponents

Chaos, as we have seen, is mainly characterized by sensitive dependence on initial conditions. Technically, this means that if I take two initial points which are very close to each other, say x_0

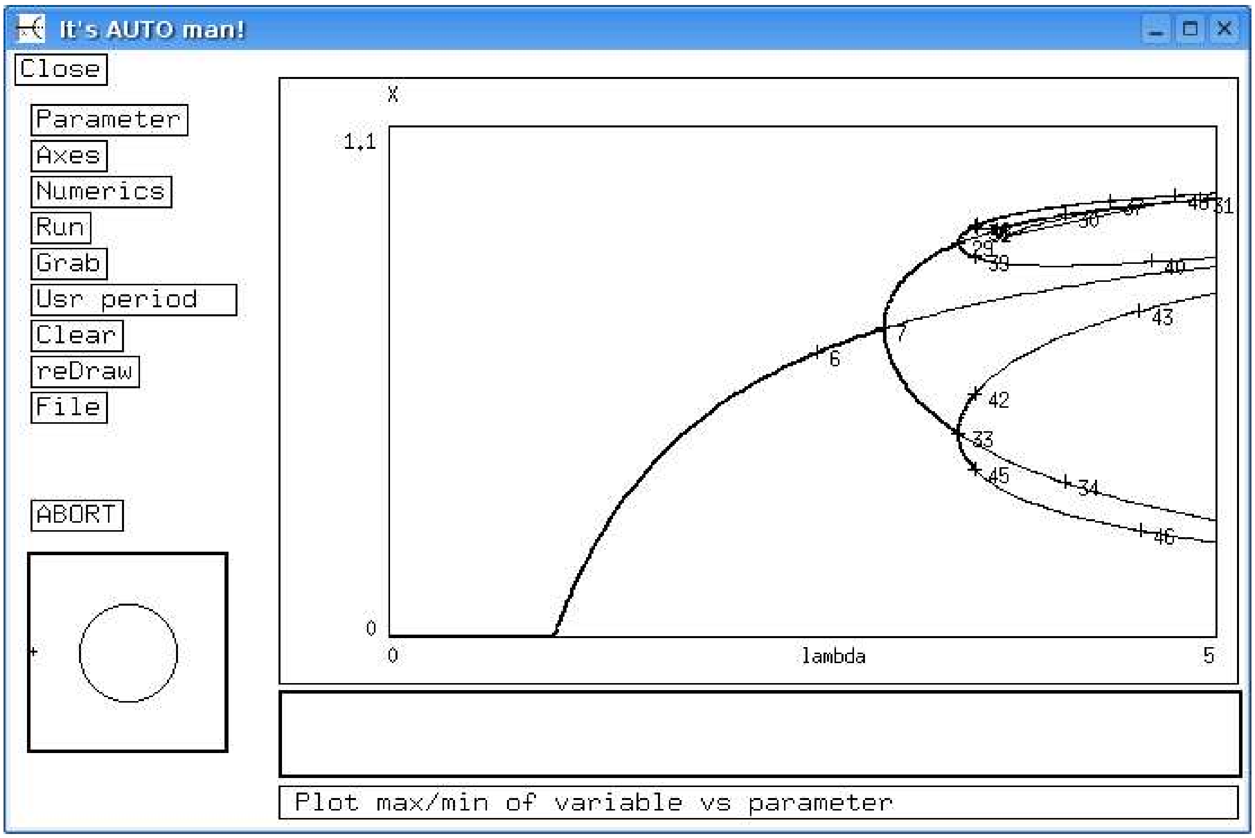


Figure 3: Bifurcation diagram of the logistic map computed using AUTO.

and $y_0 = x_0 + \delta_0$ with δ_0 very small, and iterate them under the map, then the difference between the two time series $\delta_n = y_n - x_n$ should grow exponentially. In mathematical terms,

$$|\delta_n| \sim |\delta_0| e^{\mu n}. \tag{2}$$

The constant μ is called the **Liapunov exponent**. The similarity symbol here means that this behavior is obtained for large n. Chaotic systems have at least one positive Liapunov exponent,[1] while nonchaotic systems have zero or negative Liapunov exponents.

Equation 2 can be applied to any dynamical system. For maps, this definition leads to a very simple way of measuring Liapunov exponents. Rearrange equation 2 to

$$\mu = \frac{1}{n} \ln \left| \frac{\delta_n}{\delta_0} \right|.$$

By definition,

$$\delta_n = f^{(n)}(x_0 + \delta_0) - f^{(n)}(x_0).$$

Thus,

$$\mu = \frac{1}{n} \ln \left| \frac{f^{(n)}(x_0 + \delta_0) - f^{(n)}(x_0)}{\delta_0} \right|.$$

For small values of δ_0, the quantity inside the absolute value sign is just the derivative of $f^{(n)}$ with respect to x evaluated at $x = x_0$:

$$\mu = \frac{1}{n} \ln \left| \frac{df^{(n)}}{dx} \right|_{x=x_0}.$$

Since

$$f^{(n)}(x) = f(f(f(\dots f(x)))\dots),$$

by the chain rule,

$$\begin{aligned} \left.\frac{df^{(n)}}{dx}\right|_{x=x_0} &= \left.\frac{df}{dx}\right|_{x=f^{(n-1)}(x_0)} \left.\frac{df}{dx}\right|_{x=f^{(n-2)}(x_0)} \cdots \left.\frac{df}{dx}\right|_{x=x_0} \\ &= \left.\frac{df}{dx}\right|_{x=x_{n-1}} \left.\frac{df}{dx}\right|_{x=x_{n-2}} \cdots \left.\frac{df}{dx}\right|_{x=x_0} \\ &= \prod_{i=0}^{n-1} f'(x_i). \end{aligned}$$

Our expression for the Liapunov exponent becomes

$$\mu = \frac{1}{n} \ln \left| \prod_{i=0}^{n-1} f'(x_i) \right| = \frac{1}{n} \sum_{i=0}^{n-1} \ln(|f'(x_i)|).$$

[1] In multi-dimensional systems, there will be as many Liapunov exponents as there are dimensions.

The Liapunov exponent is the large n limit of this expression, and so we have,

$$\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln(|f'(x_i)|).$$

Example 3.1 For a stable fixed point, the orbit tends to the fixed point. Since the Liapunov exponent involves an average over a trajectory, the contribution from the fixed point itself eventually dominates the average and we have

$$\mu = \ln|f'(x^*)|.$$

If the fixed point is stable, $|f'(x^*)| < 1$ so that $\mu < 0$.

Example 3.2 Computing Liapunov exponents for maps is so easy that it can be done with almost any programmable device. I used my favorite tool, Maple, to work out the Liapunov exponents for the logistic map over a range of values of λ. We start by defining the map:

```
> f := x -> lambda*x*(1-x);
```

$$f := x \to \lambda x\,(1-x)$$

We can use Maple to calculate the derivative. The following technique makes the derivative into a function, which is particularly convenient:

```
> dfdx := unapply(diff(f(x),x),x);
```

$$dfdx := x \to \lambda\,(1-x) - \lambda x$$

I like to set up the parameters of my calculation as variables in my Maple session, for two reasons: First, I can set some of the counts to low values for testing. Secondly, if I want to change something later, all the parameters are in one place and I don't have to look for them in my loops. Among other things, this avoids errors since the same number sometimes has to be used in a few different places.

We're going to set up a calculation for a few hundred different values of λ. The interesting region is from λ=3 up. We will use 1000 iterates to calculate each Liapunov exponent. Each trajectory will start from a non-rational value. Note that the use of `evalf()` in setting x_0 and in the loop that follows prevents Maple from trying to do exact arithmetic, which slows things down considerably.

```
> npts := 300;
```

$$npts := 300$$

```
> lambda_min := 3;
```

$$lambda\_min := 3$$

```
> lambda_max := 4;
```

$$lambda\_max := 4$$

```
> niter := 1000;
```

$$niter := 1000$$

```
> x0 := evalf(Pi/4);
```

$$x0 := 0.7853981635$$

The following loop does all the work. The outer loop increments the value of λ. The variable liapexp is used as an accumulator which must be reset for each new value of λ. The inner loop computes iterates of the map and the sum required for the Liapunov exponent. The rest is simple bookkeeping. Note that the final colon in the looping statements inhibits Maple from printing a lot of uninteresting intermediate output.

```
> for i from 1 to npts do
> lambda := evalf(lambda_min + (i/npts)*(lambda_max-lambda_min));
> liapexp := 0;
> x := x0;
> for j from 1 to niter do
> x := f(x);
> liapexp := liapexp + ln(abs(dfdx(x)));
> od;
> liapexp := liapexp/niter;
> if i=1 then ptlist := [lambda,liapexp];
> else ptlist := ptlist,[lambda,liapexp]; fi;
> od:
```

```
> plot([ptlist],labels=["lambda","x"]);
```

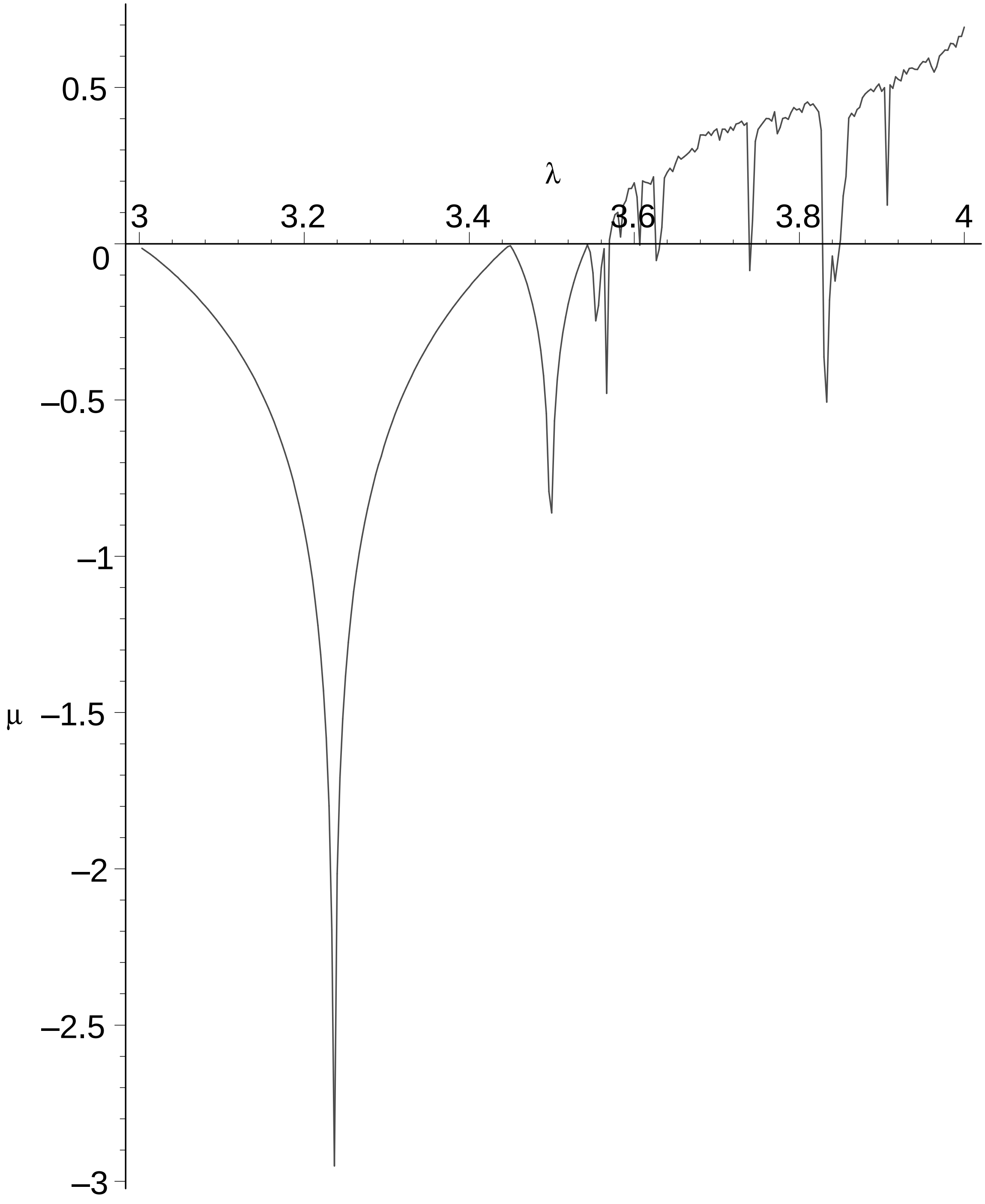


Figure 4: Liapunov exponent of the logistic map as a function of the parameter λ as calculated using the Maple routine described in the text.

> The result of this calculation is shown in Fig. 4. Note that the system passes in and out of chaos as we increase λ. The nonchaotic regions are called **periodic windows** because the attractors are periodic orbits in these regions. Periodic windows are a common feature of systems which are capable of chaotic behavior, although other types of nonchaotic solutions can also interrupt the chaotic region.

Positive Liapunov exponents are conclusive evidence of chaos, give or take some minor technical quibbles which don't come up all that often in practice. They are quite easy to calculate for maps, so there is no excuse not to do so in these cases. They are a bit more difficult to calculate for differential equations, and we won't pursue this matter further. However, if you are making a serious study of a dynamical system and claim that it is chaotic, you really *should* calculate a Liapunov exponent to prove your point.