Maps and differential equations

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Maps are algebraic rules for computing the next state of dynamical systems in discrete time. Differential equations and maps have a number of important connections which we explore in this lecture. Many of the topics mentioned in these notes are also connected to each other. You may be able to see some of the connections immediately, but in any case you should certainly think about them.

1 Numerical methods as maps

In an earlier lecture, we noted that Euler's method for integrating differential equations,

$$\mathbf{z}_{n+1} = \mathbf{z}_n + h\mathbf{f}(\mathbf{z}_n),$$

is a map: Given the state in step n, the state in step $n+1$ can be computed from the differential equation ($\mathbf{f}(\mathbf{z}_n)$) and the size of the time step h. Many numerical integration methods can be understood as maps, albeit usually more complicated ones than this.

In the limit of small step size, sensible numerical integration methods converge to a solution of the corresponding differential equation. However, at larger step sizes, there are no guarantees.

Example 1.1 Consider the logistic[1] differential equation

$$\frac{dx}{dt} = x(1-x).$$

Euler's method for this differential equation is

$$x_{n+1} = x_n + hx_n(1-x_n) = x_n(1+h-hx_n). \tag{1}$$

This is called the logistic map. (This map is actually better known in a slightly different form. Let $y_n = hx_n/(1+h)$ and $\lambda = 1+h$. Then, $y_{n+1} = \lambda y_n(1-y_n)$.)

The logistic differential equation has very simple behavior: $x = 0$ is an unstable equilibrium point, while $x = 1$ is a stable equilibrium. The logistic map on the other hand, can display an astonishing variety of attractors, depending on the parameters. Figure 1 gives some examples. For smallish values of h (up to about $h = 2$), the iterates of the logistic map 1 converge

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[1]This differential equation has applications in some problems in logistics, which is the systematic study of problems associated with the movement of goods or people to meet needs in distant locations.
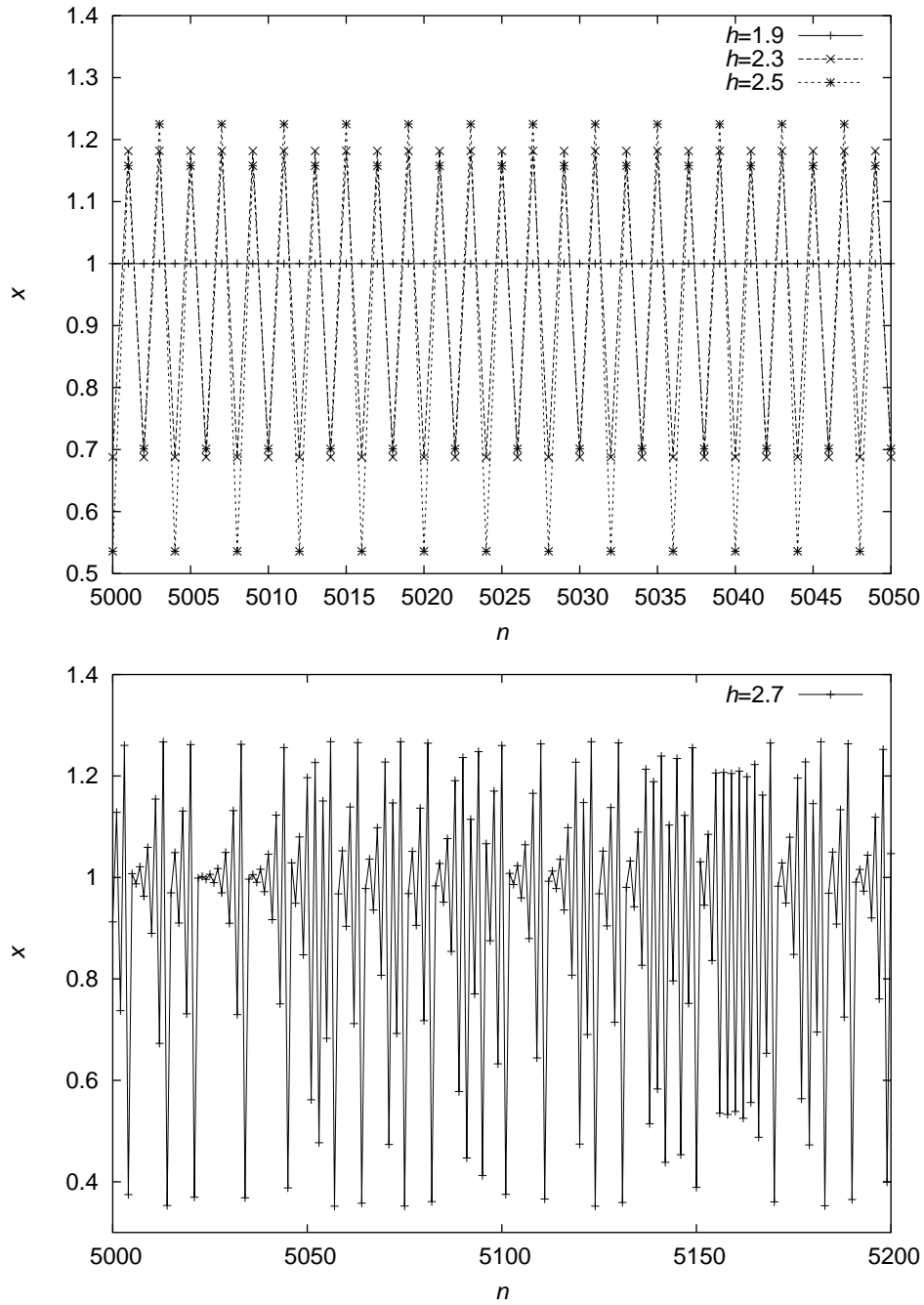
Figure 1: Some time series obtained from the application of the Euler method to the logistic differential equation for various values of h. The lines connecting the points are only there to guide the eye since values of x are only generated at integer values of n.

onto the stable equilibrium point of the logistic differential equation. However, for $h > 2$, we see periodic solutions, and eventually solutions which look chaotic. Thus, numerical methods can produce results which aren't even *qualitatively* correct. It is therefore a good idea to verify integration results carefully, varying the step size and, if possible, changing methods.

Maps don't just turn up in the integration of differential equations. Consider for instance the problem of solving the algebraic equation $f(x) = 0$ for x. One common way to do this is to use Newton's method. Expand $f(x)$ in a Taylor series near x_n:

$$f(x) = f(x_n) + (x - x_n)f'(x_n) + \ldots$$

Assuming that the first two terms are a reasonable approximation to the function and setting this expression to zero, we get

$$x_{n+1} \approx x_n - \frac{f(x_n)}{f'(x_n)}.$$

We label the solution of the truncated equation x_{n+1} in recognition of the fact that we won't get the exact answer in one step, so we'll have to iterate this procedure many times to get the correct answer, if this is going to work at all. When it doesn't work, Newton's method often generates interesting dynamics. In fact, a lot of the early work on fractals came from a study of Newton's method in the complex plane.

2   Solution maps of differential equations

It sometimes occurs that we have a differential equation for a system, but are only interested in the behavior at fixed time intervals. For instance, the logistic differential equation is sometimes used to model population growth, but we might only have census data at intervals of five or ten years. It then makes little sense to look at the whole continuous solution.

Suppose for instance that we want solutions of the logistic differential equation (*not* solutions of some numerical approximation like the Euler method iterates) at fixed intervals T. The solution of this differential equation is

```
> dsolve({diff(x(t),t)=x(t)*(1-x(t)),x(0)=x0},x(t));
```

$$\mathrm{x}(t) = \frac{1}{1 - \frac{e^{(-t)}(-1 + x0)}{x0}}$$

or

$$x(t) \quad = \quad \frac{x_0}{x_0 - e^{-t}(x_0 - 1)}. \tag{2}$$

$$\therefore e^{-t} \quad = \quad \frac{x_0(x(t) - 1)}{x(t)(x_0 - 1)}, \tag{3}$$

$$\text{and} \quad x(t+T) \quad = \quad \frac{x_0}{x_0 - e^{-(t+T)}(x_0 - 1)}. \tag{4}$$

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If we substitute for the exponential in equation 4 using equation 3 we get, after a little rearranging,

$$x(t+T) = \frac{x(t)}{x(t) - e^{-T}x_0\,(x(t)-1)}.$$ (5)

This last equation is a **solution map**. It lets us calculate $x(t+T)$ knowing only $x(t)$ and some parameters.

Unlike equation 1 which gives *approximations* to the solution of the logistic differential equation at fixed intervals h, equation 5 is *exact*. We were able to obtain this equation because we were able to solve the differential equation. In general of course, we can't do that, but we can still obtain numerical representations of the solution map by sampling the numerical solution (obtained with a good numerical method, of course) at fixed time intervals and plotting $x(t+T)$ vs $x(t)$. This is sometimes called a Ruelle plot.

3   Poincaré maps of systems with periodic nonautonomous terms

The solution maps and Ruelle plots described above are clearly related to the Poincaré sections described in the last lecture for systems with periodic driving terms. To go from a Poincaré section to a **Poincaré map**, we need only mention that, in a deterministic dynamical system, there always has to be some rule (even if we don't know what it is) that can be applied to calculate the next intersection of a trajectory with the surface of section given the last intersection. For this reason, Poincaré maps are often called **next-return maps**. Formally, let $\{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \ldots\}$ be the set of intersections of a trajectory with the surface of section. The Poincaré map is a relation $\mathcal{P}$ such that $\mathcal{P}(\mathbf{z}_n) = \mathbf{z}_{n+1}$.

In some very simple cases, Poincaré maps can be computed analytically, but this isn't usually the case. Instead, we calculate the Poincaré map numerically. In principle, if we have a d-dimensional state space (and so a $d+1$-dimensional phase space in a nonautonomous system), then the Poincaré map is a map from $\mathbb{R}^d$ into $\mathbb{R}^d$, which is hard to picture for any $d > 1$. However, it is often sufficient to use just one variable, i.e. to plot $z_i(t+T)$ vs $z_i(t)$.

Example 3.1 We return to the driven Brusselator which we studied recently:

$$
\begin{aligned}
\dot{x} &= a_0 + \alpha\sin(2\pi t/T) - bx + x^2 y - x,\\
\dot{y} &= bx - x^2 y.
\end{aligned}
$$

Suppose that we want to look at the Poincaré map using xpp. We use the xpp input file which recast the equations as follows:

$$
\begin{aligned}
\dot{x} &= a_0 + \alpha\sin(2\pi\theta/T) - bx + x^2 y - x,\\
\dot{y} &= bx - x^2 y,\\
\dot{\theta} &= 1.
\end{aligned}
$$

We set up xpp with a periodic phase space in θ of period T. We also set up a Poincaré section at some reasonable value of $\theta \in [0, T)$ and then run a trajectory with a long transient.

4

To set up the Ruelle plot, we start by doing something that doesn't on the face of it make much sense: We click on `Viewaxes→2D` and then tell `xpp` to plot y on both axes. Having done this, click on `nUmerics→rUelle plot`. Set the `x-axis shift` to 1, and leave the other two shifts at zero. This plots y_{n-1} on the x axis, and y_n on the y axis. Finally, hit `Window/zoom→Fit` to see the plot.

Fig. 2 shows two examples of Poincaré maps for the driven Brusselator. Panel (a) shows a case where we previously found quasiperiodic behavior. You will note that the Poincaré map is, like the section itself, the image of a closed curve. On the other hand, we get a different kind of figure in panel (b), at parameter values which we previously identified as being chaotic. The curve isn't continuous, and it certainly isn't closed. Note that the shape of this curve is roughly parabolic, just like the logistic map (equation 1). This isn't a coincidence, as we will see later.

The fact that the chaotic map of Fig. 2(b) is single-valued means that we can actually use it to predict the next value of y given the last value of y at the section. For instance, if $y_{n-1} = 2.2$, $y_n = 3.15$ (roughly). This means that we can compute the sequence of y's at the Poincaré section from this map. There's just one problem: If you try to do this, you will find that small errors in y_{n-1} for values near 3.2 produce large errors in y_n. In other words, the Poincaré map displays sensitive dependence on initial conditions and is chaotic.

Finally note that the reason we picked y rather than x to construct the Poincaré map is that the plot of x_n vs x_{n-1} isn't as obviously informative. Again, there's no guarantee that we will get a sensible map from one variable. It was just luck that y gave us a nice, single-valued map in the chaotic regime.

4   Poincaré sections and maps in autonomous systems

The concept of a Poincaré section or map is also useful in autonomous systems with some type of cyclic attractor. If one of the variables, say x, passes repeatedly through some particular level, we can define a Poincaré section by collecting values of the other variables every time x passes through the selected level in a particular direction, say when x is increasing. Figure 3 illustrates this procedure.

As an illustration, we return to the autocatalator, last discussed in our lectures on bifurcation theory. The equations of motion were

$$
\begin{aligned}
\dot{a} &= \mu(\kappa + c) - ab^2 - a, \\
\dot{b} &= \frac{1}{\sigma}\left(ab^2 + a - b\right), \\
\dot{c} &= \frac{1}{\delta}(b - c).
\end{aligned}
$$

Suppose that we're interested in studying the chaotic regime in this system. Figure 4 shows a projection of the attractor in the chaotic regime for this system. Note that there is hole in the middle of the attractor. We imagine a half-plane of constant b starting inside this hole which intersects the attractor to the right (going toward larger values of a). We can for instance take our section at
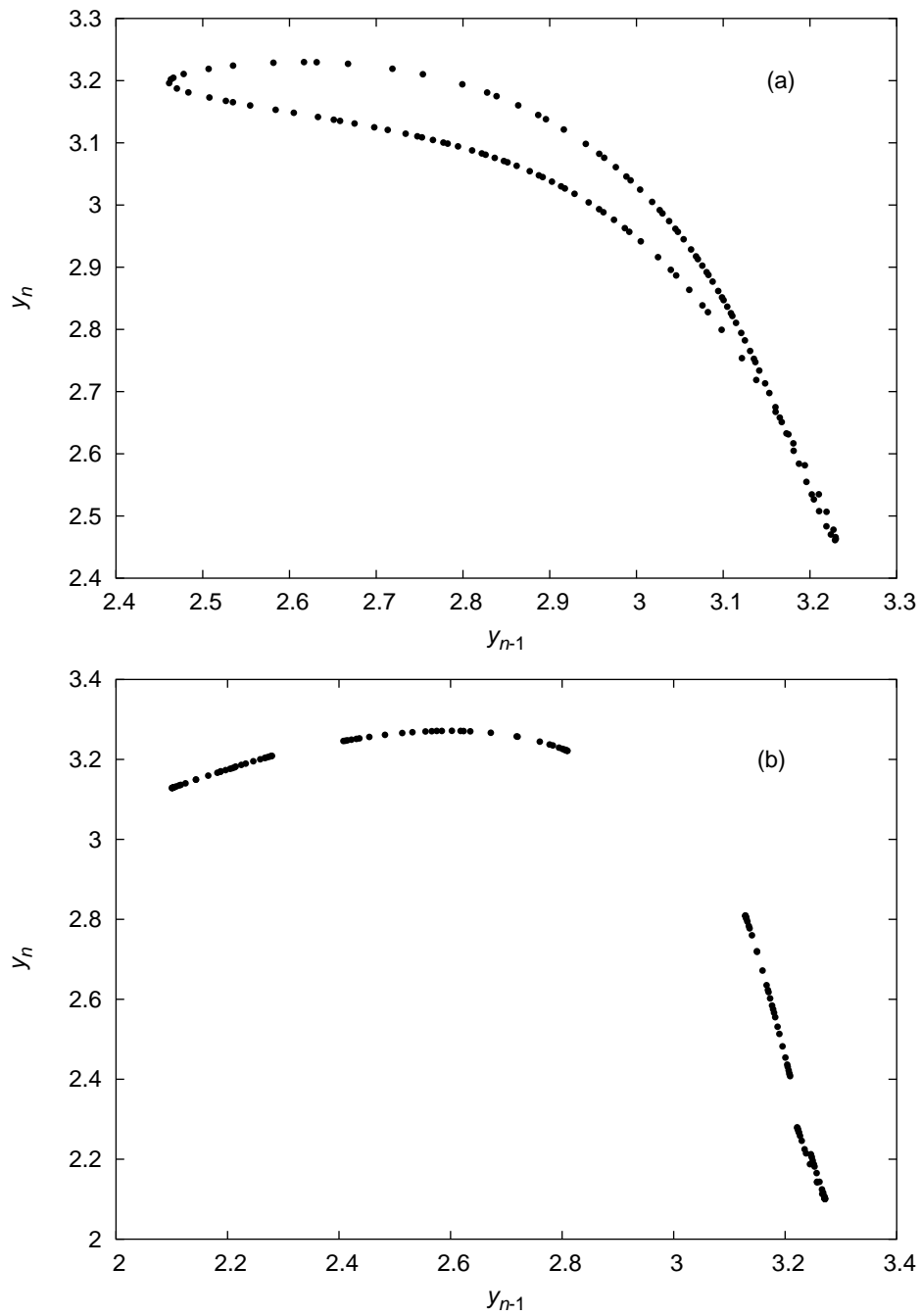
Figure 2: Poincaré maps for the driven Brusselator at parameter values $a_0 = 0.4$, $b = 1.2$, $T = 7.8$, and (a) $\alpha = 0.008$ or (b) $\alpha = 0.05$.
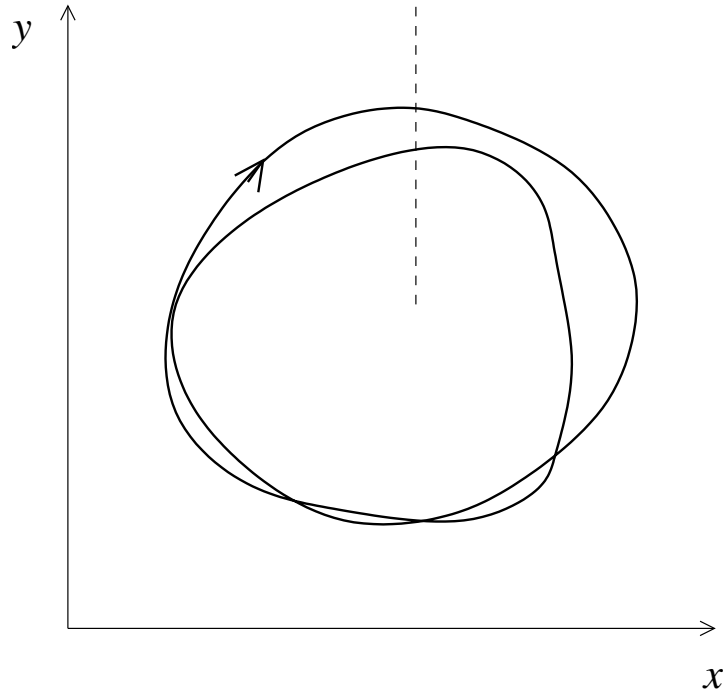
Figure 3: Computation of the Poincaré section of a system with a cyclic attractor. We pick some particular level of one of the variables, in this case x, and save the values of the other variables (e.g. y in the illustration) when x passes through the selected level in a particular direction. In this case, we decided to collect values of the other variables when x increases while passing through the section, shown as a dashed line.
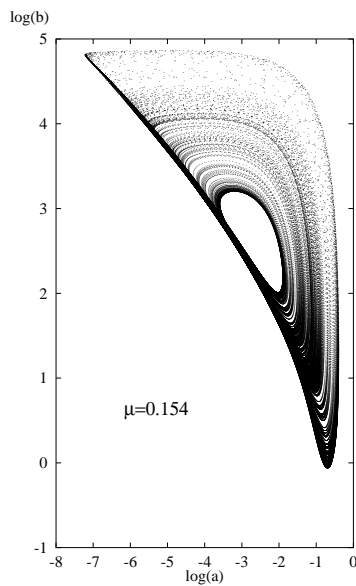

Figure 4: Strange attractor of the autocatalator at $\mu = 0.154$, $\kappa = 65$, $\delta = 0.02$ and $\sigma = 0.005$.
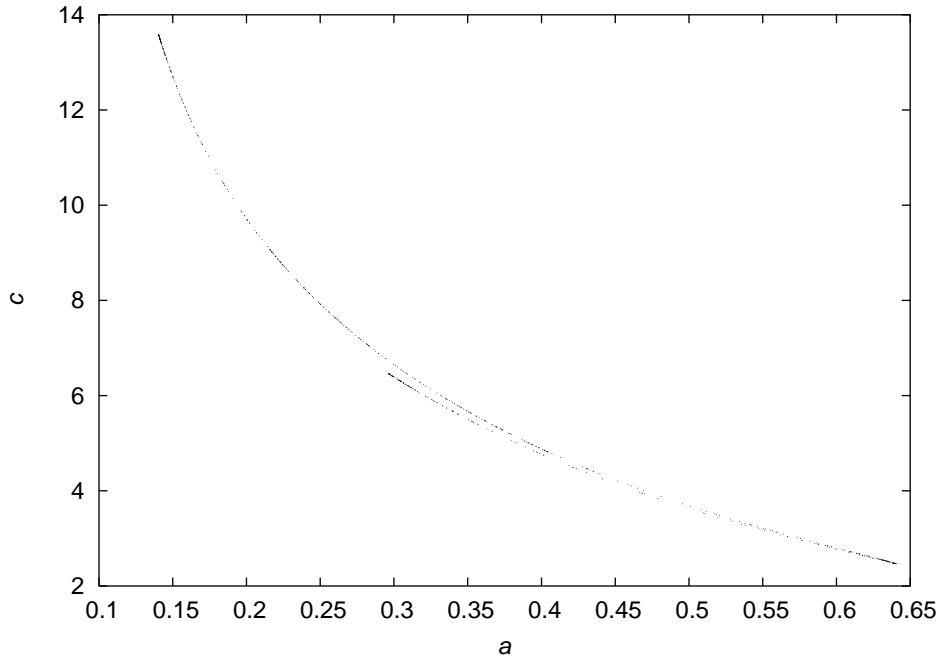
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Figure 5: Poincaré section for b increasing through the value 12 in the autocatalator at the parameters of Fig. 4.

$\log b = 2.5$, which corresponds to $b = e^{2.5} \approx 12$. We choose to cut to the right because there is a larger dispersion of values of a in this direction than in the other. This *usually* gives better Poincaré maps, although as in most other things in nonlinear dynamics, it's often necessary to experiment to get the right cut. We now need to know whether b increases or decreases in this part of the attractor. If a is large then, from the differential equation, $\dot{b} > 0$.

Having gathered all this intelligence, we're ready to go to xpp. We run the system for a little while, discarding a transient to make sure that we are on the attractor. We then set up a Poincaré section at $b = 12$, setting the Direction in the appropriate dialog box to 1, which indicates an increasing value of b. If we run the model again, we get the image shown in Fig. 5. We can compute the Poincaré map using xpp's Ruelle plot function. I decided to plot a_n vs a_{n-1}, but c would have done just as nicely. The result appears in Fig. 6. The map isn't single-valued between $a_{n-1} = 0.3$ and 0.4 (roughly), but it is pretty close to just being a simple curve. In fact, the map is pretty close to being parabolic from $a_{n-1} = 0.15$ to 0.3 or so. This again suggests that we could learn something about this system from studying analytic maps which have similar shapes.

Finally note that Poincaré sections don't have to be orthogonal to a coordinate axis and that it is sometimes convenient to use a surface of section which has some other orientation in phase space.

5   Next-amplitude maps

In some cases, we get a nicer looking map is we use a slightly different construction. Suppose that instead of collecting data on a surface of section, we collect data when a variable x reaches a
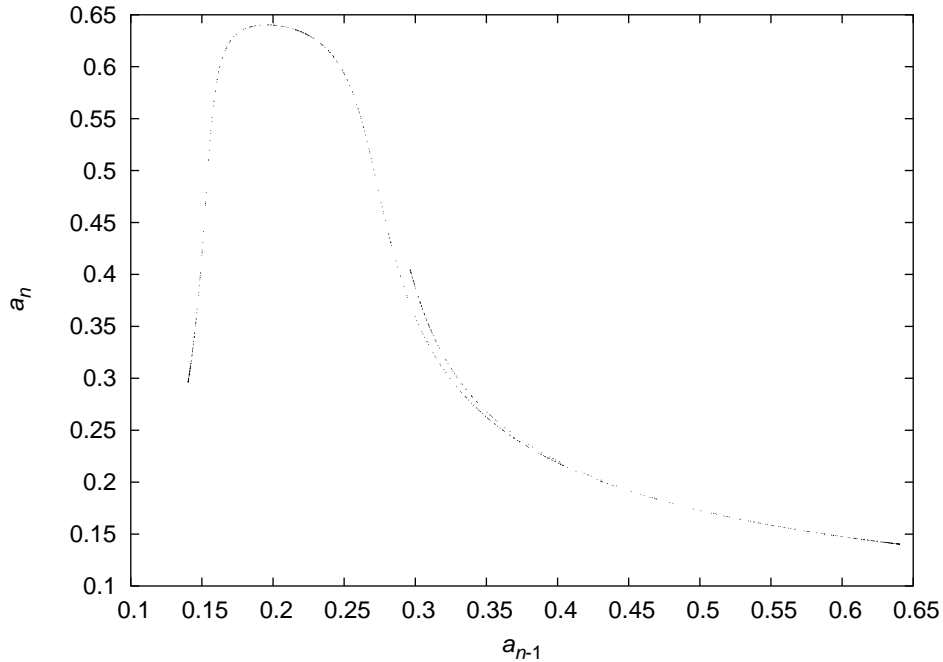
Figure 6: Poincaré map corresponding to the Poincaré section in Fig. 5.

maximum (or a minimum, for that matter). If we then plot the next maximum vs the last, we have **next-amplitude map**.

The next-amplitude map has some practical applications, but is perhaps most famous because it was one of the devices used by Edward Lorenz in his famous study of the attractor which bears his name. We retrace Lorenz's steps here.

The Lorenz equations[2] are

$$\begin{aligned} \frac{dx}{dt} &= \sigma(y-x), \\ \frac{dy}{dt} &= x(r-z)-y, \\ \frac{dz}{dt} &= xy-bz. \end{aligned}$$

These equations are a radical simplification of equations describing convection in a fluid heated from below. Integrating these equations at appropriate parameter values gives the famous Lorenz "butterfly" attractor shown in Fig. 7(a).

Since the chaotic solution shown in Fig. 7 was generated by deterministic differential equations, Lorenz thought there should be some regular trends within the data. For some reason, his attention fixed itself on the variable z. He intuited by looking at data such as those shown in Fig. 7(b) that the maximum in z might be a good predictor of what would happen on the next loop around the attractor. Based on this intuition, he constructed the first next-amplitude map, plotting one

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[2]Lorenz himself modestly calls this the "three-variable model".
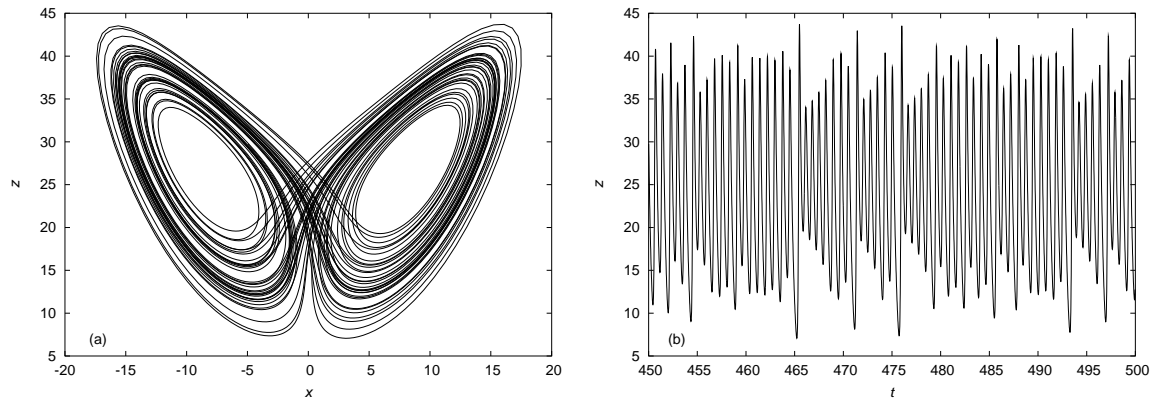
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Figure 7: Lorenz attractor (a) and time series of z vs t for $b = 2.67$, $\sigma = 10$ and $r = 28$.

maximum in z against the next. We can repeat this calculation in xpp by setting up a Poincaré section, but this time choosing the Max/min option. We set the Variable to be z. The Section value isn't used in searching for maxima, while a Direction of 1 indicates that we want maxima rather than minima (-1) or both (0). If we then set up a Ruelle plot in the usual way and plot $z_{\max,n}$ vs $z_{\max,n-1}$, we get the next-amplitude map shown in Fig. 8. In this case, the points don't exactly lie on a curve. They never do, but in this case it's obvious that the curves which make up our map have some thickness. However, we can again conclude that to a good approximation, the solutions of these differential equations define a map, this time roughly tent-shaped.

6   Concluding comments

Whether we are interested in the "dynamics of numerics", as the study of the dynamics induced by numerical methods is called, or in learning something about the chaotic solutions of certain differential equations, students of differential equations frequently end up studying maps. Maps also turn up directly as models of certain natural phenomena, particularly in population biology. In many ways, this is a fortunate thing because maps are easier to analyze than differential equations, as we will see in our next lecture.
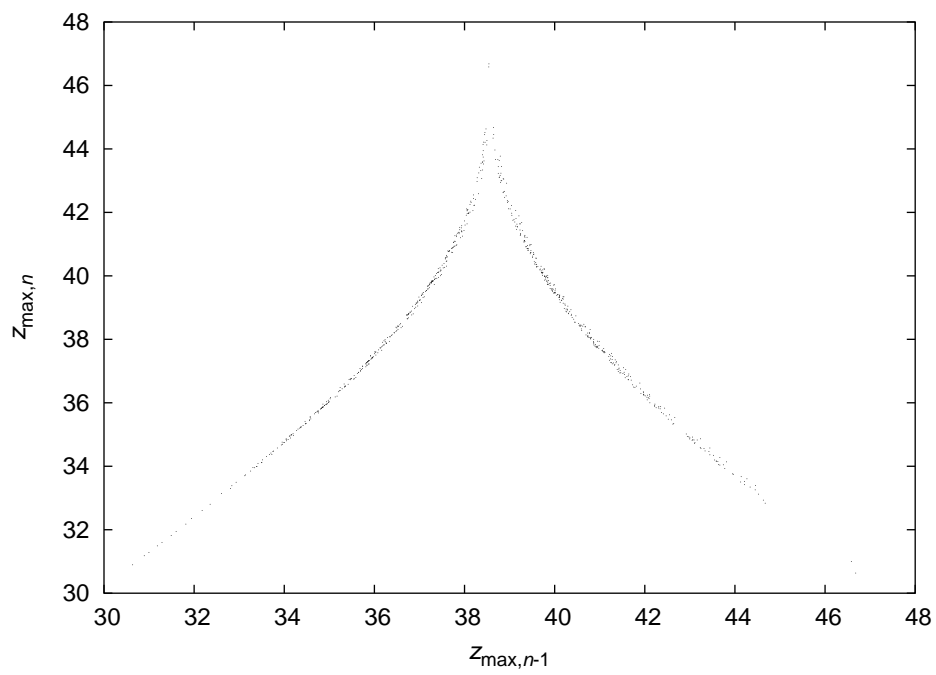
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Figure 8: Lorenz next-amplitude map for the attractor shown in Fig. 7.