Bifurcation Analysis with AUTO

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In this lecture, we will learn how to do a bifurcation analysis with the computer program AUTO. AUTO is built into xppaut, which is where the aut part of the name comes from, so we can just continue to use the tool we started to learn in the last lecture.

We will proceed by example, working with the autocatalator model we started to study last time. Because AUTO is a tricky piece of software, the instructions given in this set of notes will be much more detailed regarding the operation of the software than would otherwise be the case. Recall that the xpp input file is

```
# Autocatalator.ode

a' = mu*(kappa+c) - a*b^2 - a
b' = (a*b^2 + a - b)/sigma
c' = (b-c)/delta

param mu=0.1, kappa=65, delta=2e-2, sigma=5e-3

# The variables range over several orders of magnitude, so
# it's convenient to plot log(a), log(b) and log(c)
aux la = log(a)
aux lb = log(b)
aux lc = log(c)

# In order to avoid problems with the logs,
# start from a point other than (0,0,0).
a(0) = 1
b(0) = 1
c(0) = 1

# This system is stiff, so we need
# an appropriate integrator.
@ METHOD=stiff

# The time scale of the oscillations is really fast,
# and the spikes are really sharp and high, so we need
# to adjust both the integration step size and
# the maximum variable value allowed.
@ DT=1e-4, BOUNDS=1e4
```
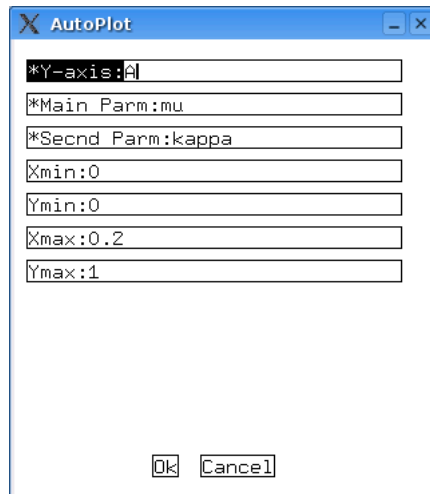
```
@ MAXSTOR=1000000

done
```

In order to use AUTO, we have to start with a known solution, preferably an equilibrium point. We know from our previous study that there is a stable focus at $\mu = 0.015$. We set μ to this value, and then get a trajectory. We then hit `Initialconds→Last` to make sure that we have gotten rid of the transient. If you do a `Window/zoom→Fit` operation, you should see a perfectly flat line. Better yet, if you look in the data viewer, the values should be perfectly constant. Now click on `File→Auto`. This will open up the AUTO window.

AUTO needs to be set up. We need to tell it what parameter(s) we will vary, what we will plot, and so on. Start by clicking on `Axes→Hi-lo`. This will set up AUTO to plot local minima and maxima of one of the components of the computed solutions as a function of a parameter. A simple limit cycle will, for instance, appear as a pair of points on our diagram. We could in principle choose any of the variables of the model to plot. I choose a.[1] Similarly, we could obtain bifurcation diagrams as a function of any of the parameters, but it makes sense to build on our earlier work and to vary μ. We need to set the scale of the bifurcation diagram. We found interesting behavior when μ was varied from 0.015 to 0.154. We didn't really look at what happens after that. It would therefore make sense to vary μ from our starting point at least up to 0.2. If we look at the graphs we generated last time, we note that $\log a$ was never larger than 0. Thus, the maximum in a itself would be less than 1. In the kind bifurcation diagram we're going to draw now, the variable goes on the Y axis and the parameter on the X axis. We therefore set up the axes as follows:

```
X  AutoPlot                      _ x

*Y-axis:A
*Main Parm:mu
*Secnd Parm:kappa
Xmin:0
Ymin:0
Xmax:0.2
Ymax:1




           Ok  Cancel
```

If all went well, the axis labels and limits in the AUTO window should reflect the values you just typed in.

We now need to set up the bifurcation diagram computation (as opposed to just the plotting axes). To do this, click on `Numerics` and set the following parameters:

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[1] Note that it's tempting to use the auxiliary variables representing $\log a$, $\log b$ or $\log c$ since, as we noted last time, the variables range over several orders of magnitude. Unfortunately, AUTO produces erratic results with auxiliary variables, so it's best to stick with the model's basic variables.

```
Par Max:  0.2
Ds:  0.002
Dsmin:  1e-5
Dsmax:  0.02
```

`Par Max` has the obvious meaning. `Ds` is a suggestion to AUTO of an appropriate amount by which to step the parameter, in our case μ. Since μ goes up to 0.2, we're using a value of `Ds` which will result in about a hundred points being used. Note that this is only a suggestion. AUTO actually adapts the step size as it goes, within the limits set by `Dsmin` and `Dsmax`. Again, given the range we want to cover, the indicated values are reasonable.

We're ready to go! Click on `Run`→`Steady state`. AUTO will start at your steady state, and attempt to follow what happens to this steady state as μ increases from its initial value (0.015) to the maximum value we set (0.2). AUTO generates two kinds of output: The computed points of the bifurcation diagram are displayed in the AUTO window, and they are printed out to the terminal. The latter looks like this:

```
BR    PT TY LAB     PAR(1)        L2-NORM         U(1)         U(2)         U(3)
 1     1 EP   1  1.500000E-02  1.486463E+00  4.999740E-01  9.898477E-01  9.898477E-01
 1     5 HB   2  1.531126E-02  1.514276E+00  4.999716E-01  1.010707E+00  1.010707E+00
 1    50      3  2.465130E-02  2.365388E+00  4.441410E-01  1.642832E+00  1.642832E+00
 1   100      4  3.485542E-02  3.339280E+00  3.605654E-01  2.347423E+00  2.347423E+00
 1   150      5  4.486207E-02  4.327713E+00  2.958103E-01  3.052998E+00  3.052998E+00
 1   200 EP   6  5.467292E-02  5.322212E+00  2.484300E-01  3.759270E+00  3.759270E+00
```

The columns of output have the following meanings:

`BR`	Branch of the bifurcation diagram
`PT`	Point number
`TY`	Type of point
`LAB`	Label used to number key points on the bifurcation diagram
`PAR(1)`	Value of the bifurcation parameter
`L2-NORM`	L_2 norm of the solution
`U(1), U(2),` etc.	Values of the variables

We'll look at some of these columns in detail later. For now, let's concentrate on just two of them, namely `TY` and `PAR(1)`. We see three types of points in this diagram:

- Most of the points don't have a type. These are "ordinary" points where the program is just continuing a branch of solutions, in this case the steady state. AUTO prints out points every so often, whether they're special or not, with the interval controlled by the numerical parameter `Npr`.

- `EP` is an end point, i.e. a place where we either started or stopped calculating the bifurcation diagram.

- `HB` is a Hopf bifurcation point.[2]

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[2]I should call this an Andronov-Hopf bifurcation point, but then AUTO's abbreviation doesn't make sense, so in this document I will revert to the older usage.
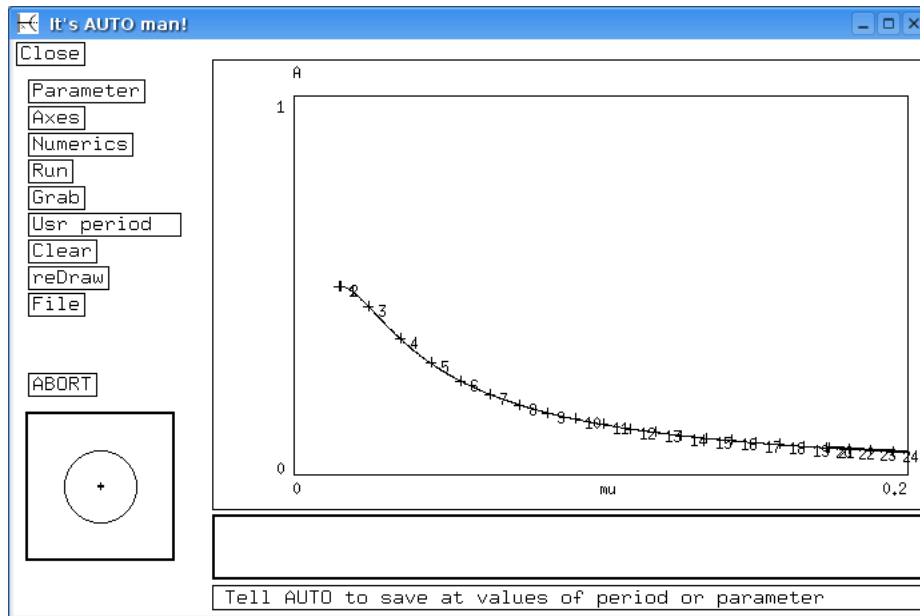
In this particular diagram, we see that a Hopf bifurcation occurs at $\mu = 1.531\,126 \times 10^{-2}$ (the value of the bifurcation parameter PAR(1) at the HB point). There's a problem though: We wanted to compute our bifurcation diagram out to $\mu = 0.2$, but the program stopped at $\mu = 5.467\,292 \times 10^{-2}$. It did this because AUTO will only compute a fixed number of points before it stops. It labeled the last point EP to indicate a normal stopping condition. We'll just have to adjust the maximum number of points and try again. Note that if there had been an error, the last point would have had type MX.

You have to be very careful when you restart AUTO to first reset the program to its original state. You do this by carrying out the following sequence of steps:

1. Click on Grab. You can move between the points using your keyboard's arrow keys. Select the first EP and hit the return key. This will reset AUTO's internal state to what it was when you started. If you get into grab mode and decide you *don't* want to grab any points, hitting the escape key will get you out without changing anything.

2. Click on File→Reset diagram. This deletes some files which AUTO uses to keep track of what it's doing.

3. Click on Clear to erase the contents of the AUTO window.

The first two steps are particularly important and must be performed *in that order*. If you mess up at this point, it will be difficult to recover, and you will have to quit AUTO and start over.

Now click on the Numerics button and increase Nmax to a large value, say 5000. (Note that the value of Nmax when you first entered this window was 200, which was the point number of the second EP.) If you click on Run, AUTO will now compute the complete bifurcation diagram. The result will look at least roughly like this:

If we look at the AUTO text output, we now see two Hopf bifurcation points, namely the one at $\mu = 1.531\,126 \times 10^{-2}$ we had seen earlier, as well as a second HB point at $\mu = 1.744\,357 \times 10^{-1}$. In

4

between these two points, the steady state is unstable. Ranges where the steady state is stable are represented by heavy lines in the AUTO graphical window, while ranges in which the steady state is unstable are shown as thin lines.

The note above that you can make mistakes from which it's hard to recover in AUTO should have sounded ominous to you. That being the case, it's a good idea to save your diagram at this point. That way, if some error occurs later, you can get back to where you were just by loading your diagram back in. Just click on File→Save diagram. The default file name should be fine, unless you want to compute several diagrams at different parameter values.

Our next task will be to see how the periodic orbits associated with the Hopf points behave. To do this, click on Grab, go the first HB point, and hit return. Then click Run→Periodic. This time, AUTO finds a lot of interesting points in a very small range of parameters (between $\mu = 1.531\,113 \times 10^{-2}$ and $1.539\,651 \times 10^{-2}$) before quitting. When it does quit, it generates the dreaded MX point type, which indicates that it ran into trouble of some sort. Before we deal with this, let's look at the new point types generated by AUTO:

- LP is a limit point. At a limit point, a branch of solutions (in this case, limit cycles) turns around. (Imagine following the upper branch of a curve that looks like ⊃. When you get all the way to the right following this curve, you turn around and come back along the lower branch.) These points are sometimes interesting, but they're not of primary interest.

- PD is a period-doubling bifurcation point.

- BP is a branch point, which is just a fancy way of saying that a bifurcation occurs here, but not one of the standard types of which AUTO happens to know the name.

In our previous study, we had missed the fact that there were bifurcations other than the Hopf bifurcation for values of μ close to this bifurcation point. This is one of the reasons that automated bifurcation analyses are useful: You can catch stuff this way you would otherwise have missed.

At this point, it's a good idea to reevaluate our strategy. We were trying to get a bifurcation diagram that covers the very large range $\mu \in [0.015, 0.2]$. We now find that there's a lot of action near $\mu = 0.015$. It might make sense at this point to start over, looking at a much smaller range of μ this time. We can cover the rest of the bifurcation diagram later.
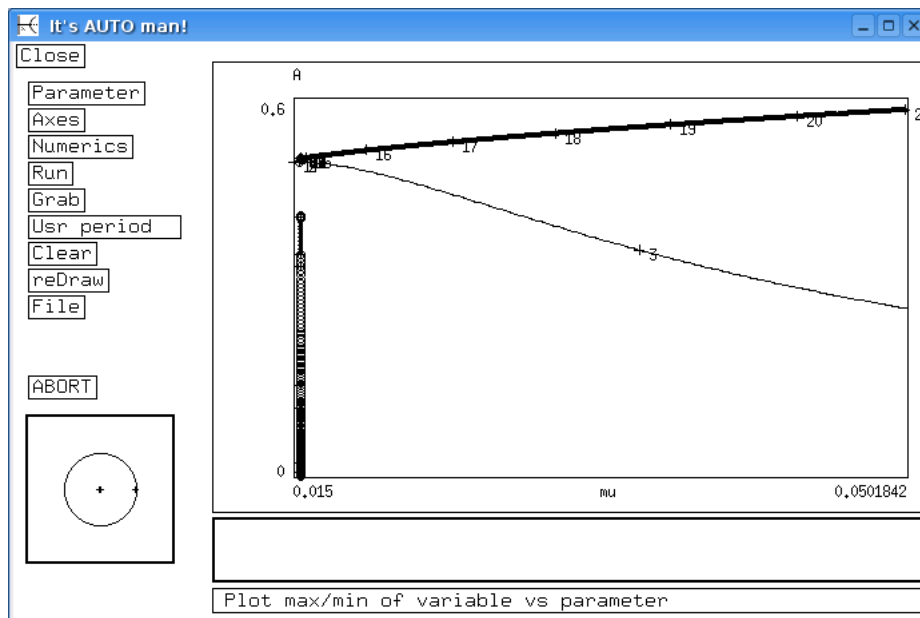
Here's my strategy: I'm going to grab the first EP (the one we started from at $\mu = 0.015$). Then I'll reset the diagram and set up AUTO to study, say, the interval $\mu \in [0.015, 0.05]$. I'll also have to worry about the fact that my last attempt bombed (error code MX). This happened while AUTO was following a branch of periodic solutions. Bard Ermentrout's[3] advice in this case is to make the numerical parameter Ntst larger. In AUTO, Ntst controls the number of mesh points used to represent a periodic orbit. Making it bigger yields a more accurate representation of the orbit at the cost of slowing the computation. The changes to the Axes dialog are obvious. The following adjustments are made to the Numerics dialog:

                    Par Max: 0.05
                    Ntst: 40

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[3]Ermentrout is the author of xppaut.

We run AUTO to get the steady-state curve in our range of interest (Run→Steady state). We then grab the HB point and click on Run→Periodic. AUTO generates the following output:

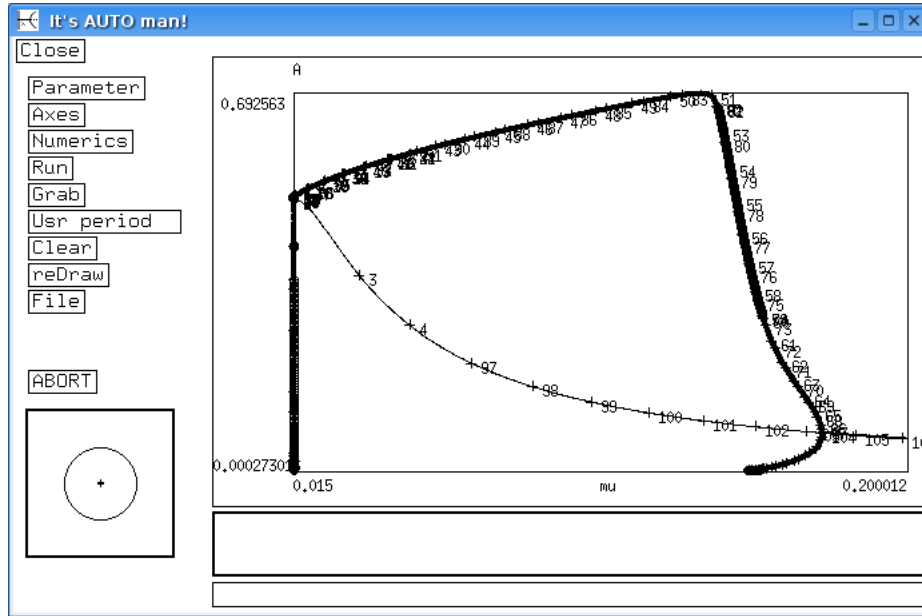| BR	PT	TY	LAB	PAR(1)	L2-NORM	MAX U(1)	MAX U(2)	MAX U(3)	PERIOD
2	3	LP	5	1.531113E-02	1.514264E+00	4.999915E-01	1.011098E+00	1.011069E+00	3.145671E-01
2	6	LP	6	1.539625E-02	1.590267E+00	5.050783E-01	1.774331E+00	1.724269E+00	6.136177E-01
2	17	BP	7	1.539626E-02	1.648621E+00	5.050643E-01	2.241218E+00	2.144174E+00	7.664522E-01
2	20	BP	8	1.539626E-02	1.669440E+00	5.050814E-01	2.409316E+00	2.291245E+00	8.084619E-01
2	52	LP	9	1.539626E-02	1.953158E+00	5.050809E-01	5.374062E+00	4.477164E+00	1.111194E+00
2	61	LP	10	1.539626E-02	2.042319E+00	5.050543E-01	6.711408E+00	5.237696E+00	1.155063E+00
2	100		11	1.539626E-02	2.459837E+00	5.050801E-01	1.643857E+01	8.501221E+00	1.236869E+00
2	137	LP	12	1.539626E-02	2.727221E+00	5.049509E-01	2.460866E+01	9.924140E+00	1.239484E+00
2	200		13	1.539626E-02	3.480340E+00	5.050394E-01	4.611810E+01	1.253354E+01	1.202768E+00
2	208	BP	14	1.539627E-02	3.601186E+00	5.050814E-01	4.914623E+01	1.289073E+01	1.194435E+00
2	300		15	1.570385E-02	5.183583E+00	5.062357E-01	7.711970E+01	1.630640E+01	9.742975E-01
2	400		16	1.917319E-02	6.723634E+00	5.177495E-01	8.478699E+01	1.733763E+01	6.614443E-01
2	500		17	2.414264E-02	8.247445E+00	5.312346E-01	8.990671E+01	1.807078E+01	4.827086E-01
2	600		18	3.002349E-02	9.777387E+00	5.448768E-01	9.433547E+01	1.874451E+01	3.729731E-01
2	700		19	3.664476E-02	1.131635E+01	5.584536E-01	9.839583E+01	1.939278E+01	3.004697E-01
2	800		20	4.391179E-02	1.286477E+01	5.718912E-01	1.022382E+02	2.001246E+01	2.498354E-01
2	879	EP	21	5.006477E-02	1.409454E+01	5.824337E-01	1.051397E+02	2.051262E+01	2.198129E-01

Note the nearly vertical section of the branch of periodic solutions, and the corresponding large number of points calculated near $\mu = 1.539\,626 \times 10^{-2}$ (points 6 to 208). Near this value of μ, the size of the limit cycle is increasing very rapidly and AUTO must do a lot of work to maintain accuracy. AUTO will sometimes get stuck in regions like this one. The best thing to do then is try to step over the problem by using larger values of Ds and Dsmin. Note also that the line of maxima which runs along the top of the page has counterpart minima along the bottom. They are so close to zero that they disappear in this plot. However, if you manually set a small negative value for Ymin in Axes→Hi-lo, these minima will reappear.

If we try to grab the first branch point, we immediately get the MX error status. This indicates that there probably isn't really a branch point here. AUTO was wrong. Following the second
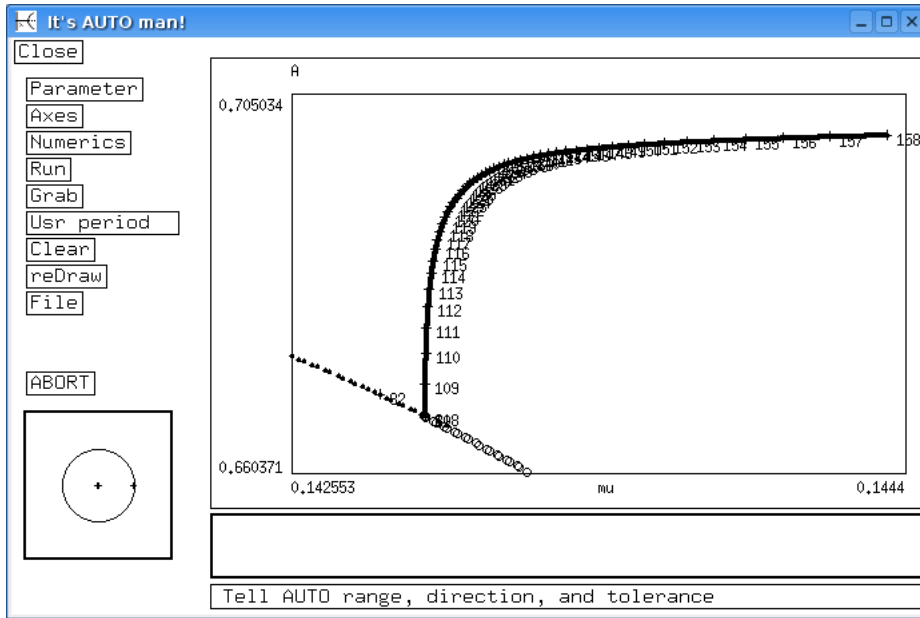
6

branch point just regenerates points we already have in our diagram. We therefore don't have to worry about this point. The same is true of the third branch point.

Since there wasn't any excitement at small μ after all, we will continue to higher values of the parameter. Set `Par max` back to 0.2 in the `Numerics` dialog. Grab the rightmost `EP` on the periodic branch we have computed. Click on `Run→Extend`. By a similar method, extend the steady-state branch. Our diagram now has the following appearance:
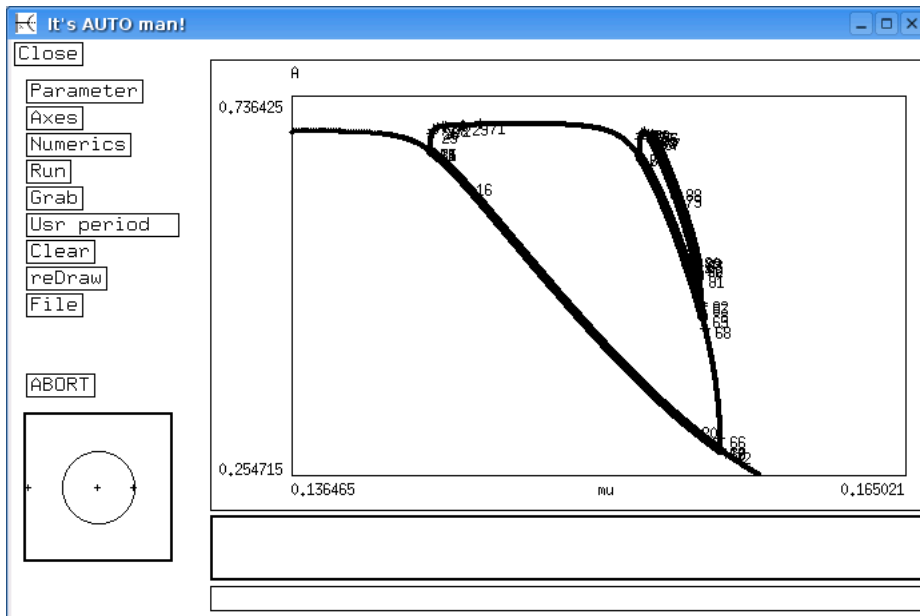

If we zoom in on parts of this diagram, we see that the periodic solutions are sometimes represented by solid dots, sometimes by open circles. The circles represent unstable periodic orbits, while the dots represent stable orbits.

Several period doubling points show up in the output. Let's follow a few of them. There's a `PD` point where the limit cycle becomes unstable at $\mu = 1.429\,566 \times 10^{-1}$. This is a likely source of a stable period-2 orbit, so let's continue this one. Grab the point and click on `Run→Doubling`. Here's a closeup of the computed piece of the doubled orbit:

We can clearly see the stable period-2 cycle emerging from the point where the period-1 cycle becomes unstable.

On the practical side, we're clearly numbering far too many points, so we'll increase `Npr` for the rest of this session. We'll also increase `Nmax`, and then extend this branch of solutions. You can click on `Abort` to stop AUTO if it seems to have filled in a part of the diagram. We again discover a period-doubling bifurcation at $\mu = 1.526725 \times 10^{-1}$ which we can continue. We can repeat this procedure until we don't seem to be getting any more detail. Here's a piece of our diagram showing the period-2, period-4 and period-8 orbits:

The period-8 orbit is so close to the period-4 that we can't actually see it in this picture.

To complete the diagram, we might as well track the steady state back to $\mu = 0$. To do this, we grab the first EP, and then set a *negative* value of the numeric parameter Ds. After a final manual scaling of the axes, we get

This is an essentially complete diagram since the higher period solutions will be invisible on the scale of this graph.

There are many other types of bifurcations than the simple Andronov-Hopf and period-doubling types seen here. AUTO can also detect **torus bifurcations** (TR). A torus bifurcation is in some ways analogous to an Andronov-Hopf bifurcation, except that it's a limit cycle that loses stability, this time to a solution which oscillates around the unstable limit cycle. The result is a trajectory which can be thought of as moving on the surface of a torus (a doughnut).