Informal definition of a dynamical system:

- Describes a system that evolves in time
- Includes
    - A time index
    - A description of the state of the system
    - A rule by which the state evolves forward in time

Formal definition:

A dynamical system is a triple $\{T, X, \varphi^t\}$, where T is a time set, X is a state space, and $\varphi^t : X \to X$ is a family of evolution operators parameterized by $t \in T$ and satisfying the following properties:

2. φ^0 is the identity operator, i.e. for $x \in X$, $\varphi^0 x = x$.

3. $\varphi^{t+s} = \varphi^t \circ \varphi^s$, i.e. for $x \in X$, $\varphi^{t+s}x = \varphi^t(\varphi^s x)$.

From Kuznetstov, *Elements of Applied Bifurcation Theory*, 2nd ed., 1998.

Dynamical systems are classified based on the properties of T, X and φ^t.

The time set T:

- Continuous or discrete?

The state space X:

- Finite or infinite?
- Continuous or discrete?
- Are objects in X finite-dimensional (like vectors) or infinite-dimensional (like functions)?

The map φ^t:

- Deterministic or stochastic?
- Autonomous or time-dependent?
- Invertible or not?

  (Some authors make a distinction between dynamical systems with invertible time-evolution and *semi-dynamical* systems for which φ^t is non-invertible, i.e. the evolution backward in time is not defined.)

- The map φ^t may also depend on some parameters. These are quantities which do not change during the evolution of the system but which may differ depending on, e.g., the experimental conditions.

Some important types of dynamical systems

- Turing machines
  - An abstract description of a computer
  - Consists of a table of instructions, an infinite tape on which binary digits are written, and a single integer register
  - Given the digit on the tape and the value held in the register, the machine looks up what to do in the table of instructions. It can write to the tape, and move the tape left or right in one move.
  - *T* is discrete.
  - *X* is discrete (consists of the possible values held in the register and of the binary number written on the tape) but infinite (since an arbitrarily large binary number can be written on the tape). It is also finite-dimensional (a vector of two integers represents the state).
  - φ^t is deterministic, autonomous (the table doesn't change) but may not be invertible.

- Finite-state automata
  - Another description of a computer
  - Similar to a Turing machine, except that the infinite tape and single register are replaced by finite memory, and that the table of instructions can contain more complicated things

- Maps
  - Models of the form $x(t+1) = f(x(t),x(t-1),\ldots)$
  - Many applications in population ecology, in the theory of differential equations, etc.
  - T is discrete.
  - X is usually continuous. It can be almost any type of space. However, when people say "map" without any additional qualifier, X is generally either the real line (or some portion thereof) or a vector of real or complex values.
  - φ^t can also be quite general.

- Ordinary differential equations
  - T is continuous.
  - X is a finite-dimensional real-valued vector.
  - φ^t is deterministic and invertible (by running time backwards). It can be autonomous or non-autonomous.

- Reaction-diffusion equations (and other related forms)
  - T is continuous.
  - X is a function space. In the simplest case, we have a single observable $u(z,t)$, where z represents the position in space. At each time t, we have to specify u at every point in the domain to fully specify the state. Thus, a function space is infinite-dimensional.
  - Parameters: rate constants, diffusion coefficients, etc.