Maps and differential equations

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What is a map?

- A map is a rule giving the evolution of a system in discrete time steps.
- General map:
$$\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n, \mathbf{x}_{n-1}, \mathbf{x}_{n-2}, \ldots)$$
- Examples:

Logistic map: $x_{n+1} = \lambda x_n(1 - x_n)$

Arnold's cat map: $\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} (2x_n + y_n) \mod 1 \\ (x_n + y_n) \mod 1 \end{bmatrix}$

Hénon map: $x_{n+1} = 1 - ax_n^2 + bx_{n-1}$.

Where do maps come from?

- The dynamics of populations that reproduce during a relatively short period of the year can often be represented by maps.
- You may recognize that numerical methods for differential equations are maps.
- For example, Euler's method is

$$\mathbf{z}_{n+1} = \mathbf{z}_n + h\mathbf{f}(\mathbf{z}_n)$$

- Maps have a number of other connections to differential equations, explored in the rest of this lecture.

Solution maps of differential equations

- Suppose that we have observations of a system at regular intervals in time, say T, and a differential equation model for the system.
- We can sometimes derive a solution map, which is to say a map that gives the solution of the differential equation at regularly spaced intervals.

Example: solution map for a second-order reaction

The second-order integrated rate law is

$$kt = \frac{1}{x(t)} - \frac{1}{x_0}$$

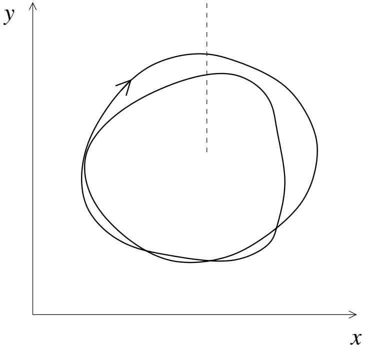
$$\begin{aligned}
\therefore k(t+T) = kt + kT &= \frac{1}{x(t+T)} - \frac{1}{x_0} \\
\therefore \frac{1}{x(t)} - \frac{1}{x_0} + kT &= \frac{1}{x(t+T)} - \frac{1}{x_0} \\
\therefore \frac{1}{x(t+T)} &= \frac{1}{x(t)} + kT
\end{aligned}$$

If we define $x(t+nT) = x_n$, then

$$x_n = \frac{1}{\frac{1}{x_{n-1}} + kT} = \frac{x_{n-1}}{1 + kTx_{n-1}}$$

Poincaré sections and maps for autonomous differential equations

- This is a technique for studying differential equations in which the solutions involve circulation around a point in phase space, including limit cycles and certain chaotic orbits.
- Imagine collecting all of the points that cross a particular surface in space in a particular direction:



Poincaré sections and maps for autonomous differential equations

- If the surface is chosen appropriately, then the points in the (Poincaré) surface of section will reveal the nature of the attractor: after decay of transients,
  - a simple limit cycle will appear as a single point
  - each period doubling will double the number of points in the section
- If $\mathbf{x}_n$ is the n'th crossing of the Poincaré section, the Poincaré map is the map relating each successive crossing, i.e. $\mathbf{x}_{n+1} = \mathbf{P}(\mathbf{x}_n)$.
- If the phase space is d-dimensional, the Poincaré surface is $d-1$-dimensional, thus $\mathbf{P}$ has $d-1$ independent components.

Example: Willamowski-Rössler model

$$\mathrm{A}_1 + \mathrm{X} \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} 2\,\mathrm{X}$$

$$\mathrm{X} + \mathrm{Y} \underset{k_{-2}}{\overset{k_2}{\rightleftharpoons}} 2\,\mathrm{Y}$$

$$\mathrm{A}_5 + \mathrm{Y} \underset{k_{-3}}{\overset{k_3}{\rightleftharpoons}} \mathrm{A}_2$$

$$\mathrm{X} + \mathrm{Z} \underset{k_{-4}}{\overset{k_4}{\rightleftharpoons}} \mathrm{A}_3$$

$$\mathrm{A}_4 + \mathrm{Z} \underset{k_{-5}}{\overset{k_5}{\rightleftharpoons}} 2\,\mathrm{Z}$$

$$\dot{x} = x(a_1 - k_{-1}x - z - y) + k_{-2}y^2 + a_3$$

$$\dot{y} = y(x - k_{-2}y - a_5) + a_2$$

$$\dot{z} = z(a_4 - x - k_{-5}z) + a_3$$

Willamowski and Rössler, *Z. Naturforsch. A* **35**, 317 (1980)

Next-amplitude maps

- In some models, a "nice" map is obtained by collecting maxima in one particular variable, and then plotting one maximum against the next one.
- This is called a next-amplitude map.