Chemistry 4010 Lecture 7: Invariant manifolds of differential equations

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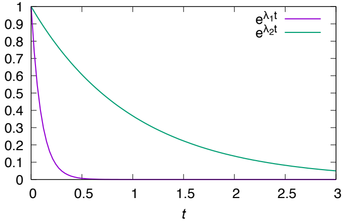
October 3, 2019

Interpreting the linearization

- Recall the linearization of a set of differential equations near an equilibrium point
$$\dot{\delta \mathbf{x}} = \mathbf{J}^* \delta \mathbf{x}$$
- Solution:
$$\delta \mathbf{x}(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t} \mathbf{u}_i$$
where $(\lambda_i, \mathbf{u}_i)$ is an eigenvector-eigenvalue pair, and n is the dimension of phase space.

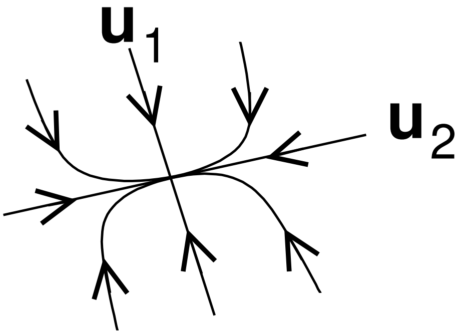
Interpreting the linearization (continued)

- Simple case: $n = 2$, both eigenvalues real and negative, and suppose that $|\lambda_1| \gg |\lambda_2|$.



- Two time scales:
  - $t \sim |\lambda_1|^{-1} \ll |\lambda_2|^{-1}$
    Over this time scale, $e^{\lambda_2 t}$ is roughly constant.
    The motion essentially consists of relaxation along $\mathbf{u}_1$.
  - $t \sim |\lambda_2|^{-1} \gg |\lambda_1|^{-1}$
    Over this time scale, $e^{\lambda_1 t}$ has become negligible.
    The motion essentially consists of relaxation along $\mathbf{u}_2$.

The effects of the linearization visualized in phase space



How points evolve in the neighborhood of a trajectory

- Suppose that we have two copies of the system, each represented by coordinates in the same phase space, say $\mathbf{x}_a$ and $\mathbf{x}_b$.
- If the differential equation is $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and we define $\Delta\mathbf{x} = \mathbf{x}_b - \mathbf{x}_a$, then

$$\begin{aligned}\dot{\Delta\mathbf{x}} &= \dot{\mathbf{x}}_b - \dot{\mathbf{x}}_a \\ &= \mathbf{f}(\mathbf{x}_b) - \mathbf{f}(\mathbf{x}_a)\end{aligned}$$

- Suppose that $\Delta\mathbf{x}$ is small, and expand $\mathbf{f}(\mathbf{x}_b)$ in a Taylor series about $\mathbf{x}_a$:

$$\begin{aligned}&\mathbf{f}(\mathbf{x}_b) \approx \mathbf{f}(\mathbf{x}_a) + \mathbf{J}(\mathbf{x}_a)\Delta\mathbf{x} \\ &\therefore \dot{\Delta\mathbf{x}} \approx \mathbf{J}(\mathbf{x}_a)\Delta\mathbf{x}\end{aligned}$$

How points evolve in the neighborhood of a trajectory
(continued)

$$\dot{\Delta \mathbf{x}} \approx \mathbf{J}(\mathbf{x}_a)\Delta \mathbf{x}$$

Observation: The relative positions of two nearby points evolving according to $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is analogous to the motion of points relative to the equilibrium point, except that we now need to look at the Jacobian near a reference trajectory.

Fact: The eigenvalues of $\mathbf{J}$ depend continuously on the parameters.

Consequence: As long as the eigenvalues are distinct with nonzero real parts, there exists a neighborhood of the equilibrium point in which the eigenvalue spectrum of $\mathbf{J}$ has a similar structure to the eigenvalue spectrum of $\mathbf{J}^*$.

Consequence: For the simple $n = 2$ system described above, $\Delta \mathbf{x}$ first shrinks rapidly along $\mathbf{u}_1(\mathbf{x})$, then more slowly along $\mathbf{u}_2(\mathbf{x})$.

Invariant manifolds

- Suppose that we have a trajectory that enters the equilibrium point (or exits from it) along a particular eigenvector.
- Within some neighborhood of the equilibrium point, the motion relative to that trajectory is analogous to the motion near the equilibrium point.
- In some sense then, such a trajectory is an extension of the corresponding eigenvector into the region where nonlinearity may be important.
- We can generalize this idea to groups of trajectories that extend an eigenspace, i.e. space spanned by a set of eigenvectors chosen in some particular way.
- Such objects are called invariant manifolds.

Definitions

Differentiable manifold: a continuously and smoothly parameterizable geometric object

Parameterizability: at any point in a d-dimensional manifold embedded in an n-dimensional space, we can write

$$\mathbf{z} = \mathbf{h}(\mathbf{y})$$

for some function $\mathbf{h}$, where $\mathbf{y}$ is a set of d coordinates and $\mathbf{z}$ is the set of the remaining $n - d$ coordinates.
The selection of $\mathbf{y}$ and the form of the function $\mathbf{h}$ may vary in different regions of the manifold, but it is possible to stitch these local representations together continuously and smoothly.

Definitions

Invariant set: any set of points that are mapped into points in the same set by the time evolution operator

If $\mathcal{I}$ is an invariant set, then $\varphi^t(\mathcal{I}) = \mathcal{I}$ for any t.

Invariant manifold: an invariant set of a dynamical system that is also a differentiable manifold

Examples of invariant sets

(and of things that aren't)

- An equilibrium point is an invariant set since $\varphi^t(\mathbf{x}^*) = \mathbf{x}^*$.
- A set of isolated equilibrium points (e.g. the three equilibria of a bistable system) is an invariant set $\mathcal{I} = \{\mathbf{x}^*, \mathbf{x}^\dagger, \ldots\}$ since $\varphi^t(\mathcal{I}) = \mathcal{I}$.
- A point that is not an equilibrium is not an invariant set since $\varphi^t(\mathbf{x}) \neq \mathbf{x}$ for any (or most) $t \neq 0$.
- The trajectory through a given point obtained by integrating to $\pm\infty$ from that point is an invariant set.

  Let $\Phi^+(\mathbf{x_0}) = \{\varphi^t(\mathbf{x_0}) \forall t \in [0..\infty)\}$ and $\Phi^-(\mathbf{x_0}) = \{\varphi^t(\mathbf{x_0}) \forall t \in [0..-\infty)\}$. Then $\mathcal{I} = \bigcup(\Phi^+(\mathbf{x_0}), \Phi^-(\mathbf{x_0}))$ is an invariant set.
- Any set of trajectories extended to $t = \pm\infty$ is an invariant set.

Examples of invariant manifolds

(and of things that aren't)

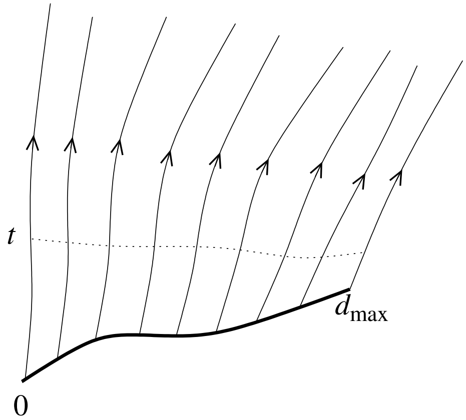
- A single equilibrium point is an invariant manifold with $d = 0$ with the trivial parameterization

$$\mathbf{x} = \mathbf{z} = \mathbf{x}^*$$

- A set of two or more isolated equilibrium points is not an invariant manifold because it lacks a smooth parameterization.

- A trajectory extended to $t = \pm\infty$, $\mathcal{T}$, is a one-dimensional invariant manifold: it's a smooth curve in space, and for every $\mathbf{x} \in \mathcal{T}$ there exists an $\mathbf{x}_0 \in \mathcal{T}$ such that $\varphi^t \mathbf{x}_0 = \mathbf{x}$ for any given value of t.

How to make a two-dimensional invariant manifold



- Integrate both forward and backward in time from "origin line"
- Parameterize this manifold by distance along origin line and time needed to reach a point along a trajectory from the origin line

Special eigenspaces from the linearization

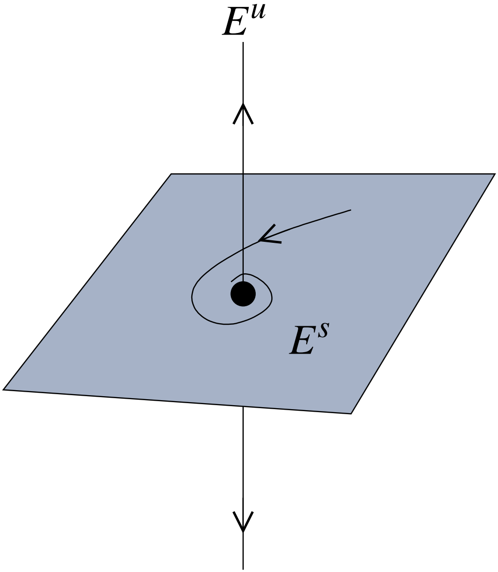
Stable eigenspace: E^s is spanned by the eigenvectors whose corresponding eigenvalues have negative real parts

Unstable eigenspace: E^u is spanned by the eigenvectors whose corresponding eigenvalues have positive real parts

Center eigenspace: E^c is spanned by the eigenvectors whose corresponding eigenvalues have zero real parts

Note: In the case that we have a pair of complex-conjugate eigenvalues, the corresponding eigenspace is spanned by the real and imaginary parts of these eigenvalues, i.e. it is two-dimensional.

Example of an eigenspace decomposition



Stable and unstable manifolds

The stable manifold ($\mathcal{M}^s$) of an equilibrium point $\mathcal{P}$ is the set of points in phase space with the following two properties:

1. For $\mathbf{x} \in \mathcal{M}^s$, $\varphi^t(\mathbf{x}) \to \mathcal{P}$ as $t \to \infty$.
2. $\mathcal{M}^s$ is tangent to E^s at $\mathcal{P}$.

The unstable manifold ($\mathcal{M}^u$) of an equilibrium point is the set of points in phase space with the following two properties:

1. For $\mathbf{x} \in \mathcal{M}^u$, $\varphi^t(\mathbf{x}) \to \mathcal{P}$ as $t \to -\infty$.
2. $\mathcal{M}^u$ is tangent to E^u at $\mathcal{P}$.

Note: These, and other similarly defined manifolds, are necessarily both invariant and differentiable.

The center manifold and its theorem

The center manifold of an equilibrium point $\mathcal{P}$ is an invariant manifold with the added property that the manifold is tangent to E^c at $\mathcal{P}$.

Note: the center manifold is not uniquely defined.

Center-manifold theorem: In some neighborhood U of an equilibrium point with stable and centre eigenspaces, but no unstable eigenspace, there exists a unique centre manifold $\mathcal{M}^c$ such that, for any $\mathbf{x} \in U$, $\varphi^t(\mathbf{x}) \to \mathcal{M}^c$ as $t \to \infty$.

Consequence: we can finally figure out the stability of an equilibrium with center and stable eigenspaces by looking at what happens in the center manifold.