# Chemistry 4010 Lecture 3: Lyapunov functions

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September 8, 2019

# Local vs global stability

- Linear stability analysis tells us about the local stability of an equilibrium point, i.e. what happens close to that point, but it tells us nothing about what happens farther away.
- We often want to know about the global stability of an attractor (e.g. an equilibrium point).
- In favorable cases, we can determine the global stability from phase-plane analysis, but of course that is only helpful for two-dimensional systems.

# Lyapunov functions

- Suppose that we have a system of ODEs $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$.
- Now suppose that we have a function $V(\mathbf{x})$.
- Since $V$ depends on $\mathbf{x}$ and $\mathbf{x}$ changes with time, the value of $V$ also changes with time.
- Chain rule:

$$\frac{dV}{dt} = \sum_i \frac{\partial V}{\partial x_i} \frac{dx_i}{dt} = \sum_i \frac{\partial V}{\partial x_i} f_i(\mathbf{x})$$

# Lyapunov functions

Definition: $V$ is a positive-definite function for an equilibrium point $\mathbf{x}^*$ on a region of phase space $U$ containing $\mathbf{x}^*$ if

1. $V(\mathbf{x}^*) = 0$, and
2. $V(\mathbf{x}) > 0$ for any $\mathbf{x} \in U - \{\mathbf{x}^*\}$.

Theorem: Given a positive-definite function for $\mathbf{x}^*$, $V(\mathbf{x})$, all solutions in $U$ tend asymptotically to $\mathbf{x}^*$ if $\dot{V}(\mathbf{x}) < 0$ for all $\mathbf{x} \in U - \{\mathbf{x}^*\}$.

A function $V$ satisfying the conditions of the theorem is called a Lyapunov function.

Implication: If we can find a Lyapunov function, then we have proven stability of the equilibrium point in the region $U$, which could be the entire physically realizable domain.