Sample solution for a test question using XPPAUT

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September 30, 2019

Question: Carry out a numerical bifucation analysis of the following predator-prey model using AUTO:

$$\dot{C} = kC\frac{a * R^2}{1 + aT_hR^2} - dC,$$
$$\dot{R} = gR(1 - R/Q) - C\frac{aR^2}{1 + aT_hR^2}.$$

The predator population is C, and the prey population is R. Vary the value of k. Use the following parameters: $a = 0.002$, $T_h = 4$, $d = 0.1$, $g = 0.5$, $Q = 100$.

Hint: What happens if $k = 0$? This might give you a good starting point for running AUTO.

Text in italics contains explanations that would not typically be included in your answer.

Answer: If $k = 0$, the equilibrium predator population is $C = 0$. The prey population then obeys the logistic equation. We know that the stable equilibrium point for the logistic equation is $R = Q$. We can therefore start AUTO from a stable equilibrium point if we set $k = 0$ and use $(C, R) = (0, Q)$ as the initial conditions. My XPPAUT input file therefore includes the differential equations, the parameter definitions given above with $k = 0$, and the initial conditions

```
C(0)=0
R(0)=100
```

Because we are starting at an equilibrium point, we can go straight to running AUTO. *I set it up to show R vs k. (C vs k would have been OK, too.) I didn't play around with the parameters too much, and ran it to see what I would get.* My first bifurcation diagram is shown in Fig. 1. There is a transcritical bifurcation *(denoted `BP` in* AUTO*)* at $k = 4.05$.

The nearly vertical unstable branch running to values of $R > Q$ is of no physical interest. *We could try to continue the $R = Q, C = 0$ unstable branch to see if it regains stability, but by experience I can tell you that this is essentially never the case when a branch that runs right along the edge of the physically realizable regime loses stability.* I am therefore going to continue the stable branch of equilibria that runs downward in my figure. *(It would run upward in a bifurcation diagram showing C vs k.) The reason I am going to continue this branch and not just be satisfied with finding a transcritical bifurcation is that the value of R is changing very rapidly in this range of parameters. It can't run down past zero, so at the very least I expect this branch to take a sharp curve sometime. I should investigate that. Will this branch run right down to zero? Or will it level off somewhere?* When I continue the stable branch, I find that it becomes unstable, and then regains stability in a pair of Andronov-Hopf bifurcations. If I then compute the branch of periodic solutions from one of the Andronov-Hopf bifurcation points, I get the bifurcation diagram shown in Fig. 2. The Andronov-Hopf bifurcations occur at $k = 0.422$ and 0.696. The creation of a stable limit cycle when an equilibrium point loses stability through an Andronov-Hopf bifurcation means that these bifurcations are supercritical.

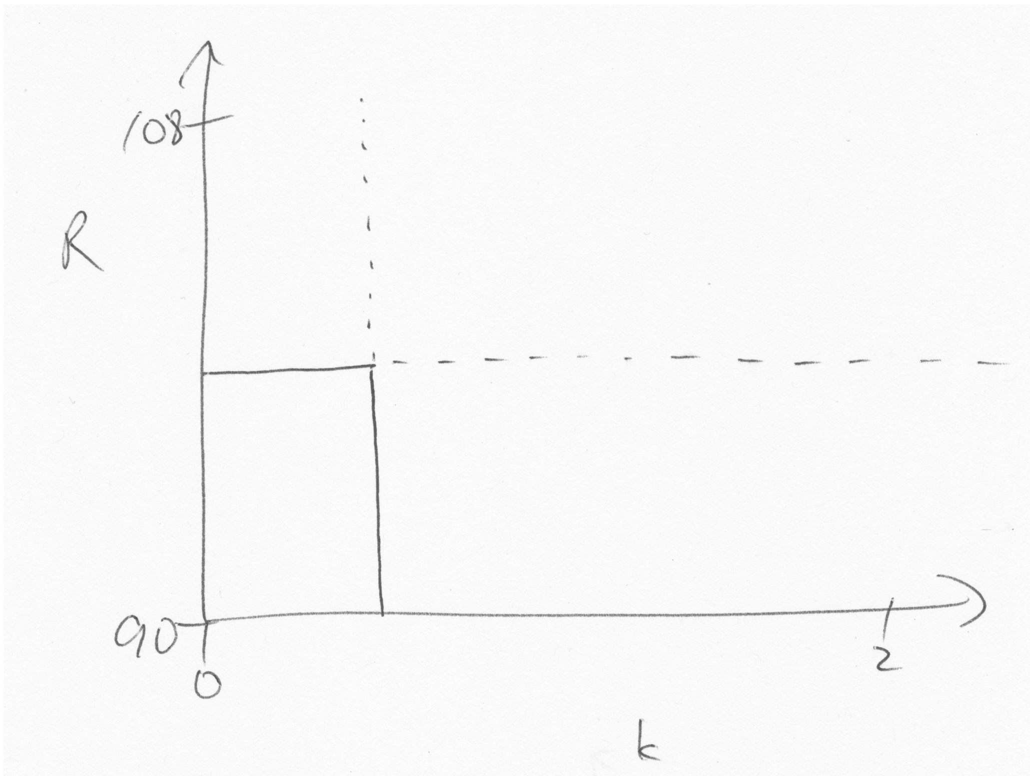

Figure 1: First try at a bifurcation diagram for the predator-prey model. Solid lines denote stable equilibria, while dashed lines denote unstable equilibria. *Note that the axes are fully labeled. This is important since I need to be able to get a sense of whether you carried out the computation correctly. The drawing isn't a perfect reproduction, but it's reasonably faithful to what you would see on your screen if you were to do this calculation.*

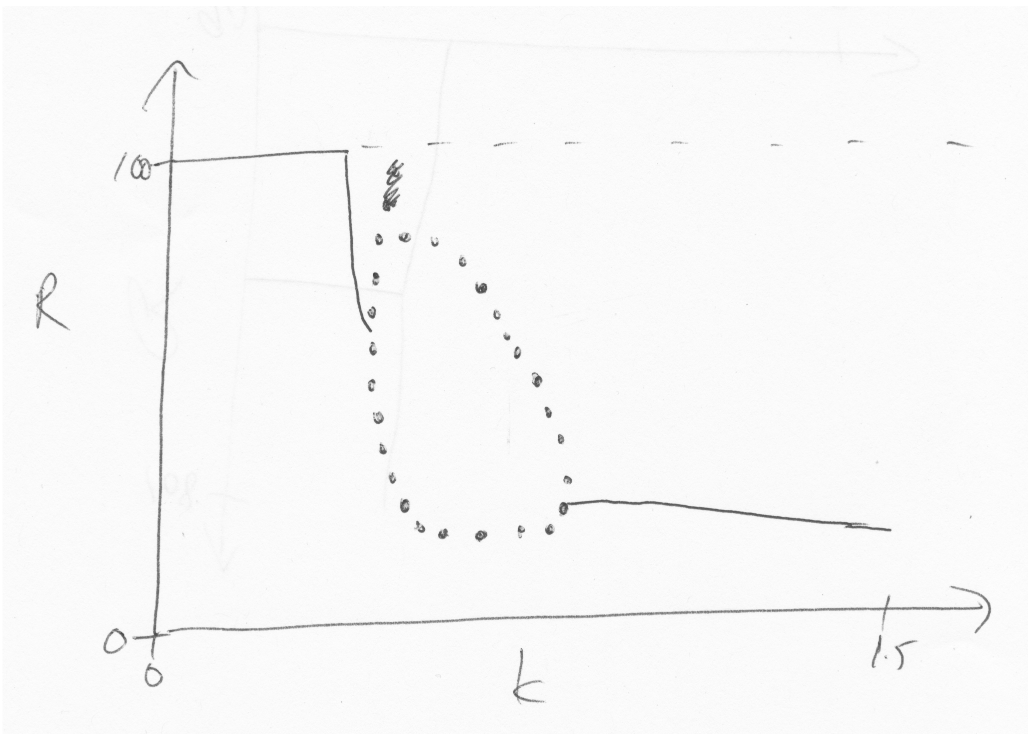

Figure 2: Complete bifurcation diagram for the predator-prey model. The dots represent stable limit cycles.