Analysis of an SI model

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We are studying a model for an infectious disease with 100% mortality. The equations are

$$\frac{dS}{dt} = rS\left(1 - \frac{S}{K}\right) - cIS,$$
$$\frac{dI}{dt} = cIS - mI.$$

Transform to dimensionless form using the following definitions:

$$s = S/K, \qquad y = cI/m, \qquad \tau = rt,$$
$$\alpha = m/r, \qquad \beta = cK/r.$$

The dimensionless rate equations are

$$\dot{s} = s(1 - s) - \alpha ys,$$
$$\dot{y} = \beta ys - \alpha y.$$

We look for equilibria by solving

$$s(1 - s) - \alpha ys = 0,$$
$$\beta ys - \alpha y = 0.$$

There are three solutions:

$$(s^\dagger, y^\dagger) = (0, 0), \qquad \text{(extinction)}$$
$$(s^\ddagger, y^\ddagger) = (1, 0), \qquad \text{(disease dies out)}$$
$$(s^*, y^*) = \left(\frac{\alpha}{\beta}, \frac{\beta - \alpha}{\alpha\beta}\right) \qquad \text{(disease is endemic)}$$

For the stability analysis, we need the Jacobian:

$$J = \begin{bmatrix} 1 - 2s - \alpha y & -\alpha s \\ \beta y & \beta s - \alpha \end{bmatrix}.$$

At $(s^\dagger, y^\dagger)$, we have

$$J^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & -\alpha \end{bmatrix}.$$

This matrix has eigenvalues 1 and $-\alpha$, therefore the extinction equilibrium is **unstable**.

At $(s^\ddagger, y^\ddagger)$,

$$J^\ddagger = \begin{bmatrix} -1 & -\alpha \\ 0 & \beta - \alpha \end{bmatrix}.$$

The characteristic polynomial is

$$|\lambda I - J^\ddagger| = (\lambda + 1)(\lambda - \beta + \alpha),$$

so the eigenvalues are -1 and $\beta - \alpha$. If $\beta > \alpha$, then this point is unstable. On the other hand, if $\alpha > \beta$, this point is stable. From the definitions of the parameters, the condition for stability is $m > cK$. Thus, the disease dies out if mortality is high compared to transmission.

At the last equilibrium point, (s^*, y^*), we have

$$J^* = \begin{bmatrix} -\frac{\alpha}{\beta} & -\frac{\alpha^2}{\beta} \\ \frac{\beta-\alpha}{\alpha} & 0 \end{bmatrix}.$$

The characteristic polynomial is

$$\lambda^2 + \frac{\alpha}{\beta}\lambda + \frac{\alpha(\beta - \alpha)}{\beta} = 0.$$

Think of this as

$$\lambda^2 + c_1\lambda + c_0 = 0.$$

$c_1 > 0$. The sign of c_0 on the other hand is not fixed. Now consider

$$\lambda = \frac{1}{2}\left\{-c_1 \pm \sqrt{c_1^2 - 4c_0}\right\}.$$

If $c_0 > 0$, then the quantity under the square root is either positive and smaller than c_1^2, or it is negative. Either way, the real parts of both eigenvalues are negative. If, on the other hand, $c_0 < 0$, then the quantity under the square root is greater than c_1^2, and $\lambda_+ > 0$. We conclude that (s^*, y^*) is stable if $\beta > \alpha$ and unstable if $\alpha > \beta$. This is precisely the reverse of the situation at $(s^\ddagger, y^\ddagger)$. Thus, the disease becomes endemic is mortality is relatively smaller than transmission. Moreover, consider the case $\alpha = \beta$. Then $(s^*, y^*) = (1, 0) = (s^\ddagger, y^\ddagger)$. The two equilibrium points exchange stability when they pass through each other, which by definition is a **transcritical bifurcation**.

Summary:

	$(s^\ddagger, y^\ddagger)$ (disease dies out)	(s^*, y^*) (disease endemic)
$\alpha > \beta$	stable	unstable
$\alpha < \beta$	unstable	stable