Modelling Biochemical Reaction Networks

Lecture 19: Introduction to bifurcations

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Recommended reading

- Fall, Marland, Wagner and Tyson, sections A.4 and A.5

Phase space

- Many biochemical models take the form of autonomous (no explicit dependence of right-hand side on time) ordinary differential equations.

$$\frac{dx_i}{dt} = f_i(\mathbf{x}), \qquad i = 1, 2, \ldots n$$

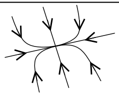
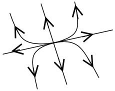
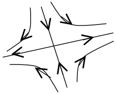
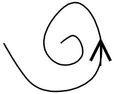
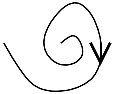
Phase space: space of independent variables (x_i) of a system. The phase-space variables define the state of the system: knowing the coordinates of a system in phase space fully defines its future evolution.

Analogy: Studying the trajectories of a system in phase space is analogous to looking at planetary orbits: There is an implied time dependence, but the shapes of the orbits can be described without talking about time.

Behavior near a steady state

- We can classify steady states according to the behavior of trajectories near these points in phase space.
- It is sufficient to look at steady states in a two-dimensional phase space (a.k.a. phase plane). Steady states in higher-dimensional spaces can be described in similar terms.

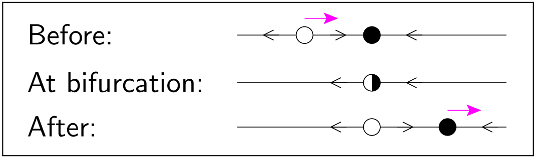
Classification of steady states

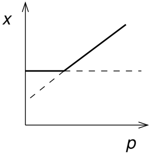
Type	**Cartoon**
Stable node	
Unstable node	
Saddle point	
Stable focus	
Unstable focus	

Local bifurcations

- A bifurcation is a qualitative change in the behavior of a model as parameters are changed.
- A local bifurcation involves changes in the number and/or types of steady states.
- Often illustrated using cartoons in which a filled dot ($\bullet$) represents a stable steady state and an open circle ($\circ$) represents an unstable steady state.
- Some of the simpler bifurcations can be observed in systems with a one-dimensional phase space.
- Any bifurcation that can occur in a d-dimensional phase space can also occur in a $(d + 1)$-dimensional phase space.

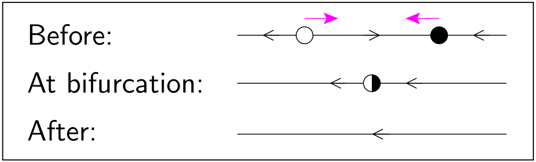
Transcritical bifurcation

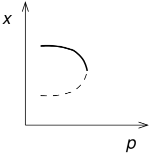




- ◑ represents a semi-stable point, in this case stable from the right and unstable from the left.
- In chemical (including biochemical) and ecological models, the immobile steady state is often at $x = 0$ (extinction/washout).

Saddle-node bifurcation

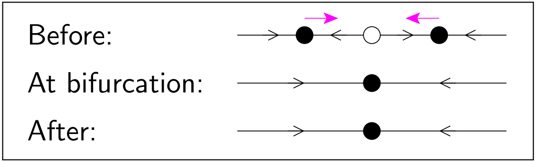


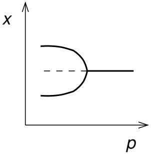


- In a two- or higher-dimensional phase space, the unstable point is a saddle, and the stable point is a node.
- The bistability studied in our two-variable model of the cell cycle is associated with a pair of saddle-node bifurcations.

Pitchfork bifurcation

Supercritical

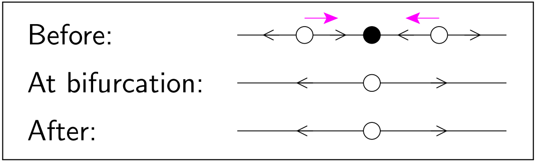


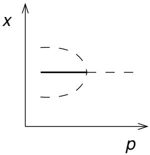


- This is another way to get bistability.

Pitchfork bifurcation

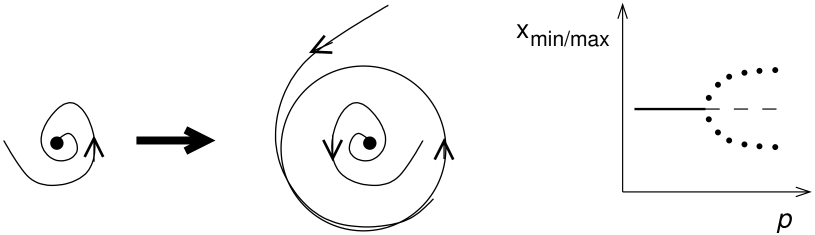
Subcritical





Andronov-Hopf bifurcation

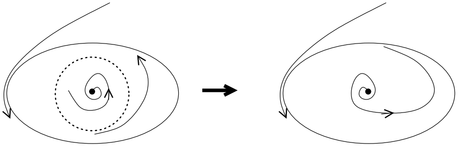
Supercritical

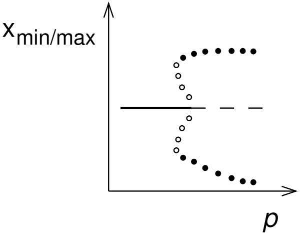


- Also known as a Hopf or Poincaré-Andronov-Hopf bifurcation.
- Creates a stable limit cycle (filled circles), an oscillatory solution of fixed amplitude and period (for fixed values of the parameters) reached from any initial conditions within its basin of attraction.
- The limit cycle has zero amplitude at the bifurcation and "grows out" of the steady state.

Andronov-Hopf bifurcation

Subcritical





- An unstable limit cycle (open circles) is created going backwards from the bifurcation value of the parameter.
- Going forwards, the system suddenly starts to oscillate with large amplitude.
- Occurs in our four-variable model of the cell cycle