Chemistry 2720 Fall 2001 Assignment 8 Solutions

1. (a) $v = \frac{d}{t} = \frac{10.00\times10^{-6}\,\mathrm{m}}{250\,\mathrm{s}} = 4\times10^{-8}\,\mathrm{m/s}$

   (b) The time is measured accurately so $\Delta v = \frac{1}{t}\Delta d$. Since the resolution of the microscope is $0.5\,\mu$m, we certainly can't determine the distance travelled any more accurately than that, even though the rulings on the micrometer are more etched with exquisite precision. In fact, since there is a similar uncertainty in determining when the spore has passed both the start and finish lines, Δd should be twice this figure, i.e. $1\,\mu mathrmm$.[1] Thus, $\Delta d = 1\times10^{-6}\,\mathrm{m}$ and $\Delta v = \frac{1}{250\,\mathrm{s}}(1\times10^{-6}\,\mathrm{m} = 4\times10^{-9}\,\mathrm{m/s}$.

   (c) The uncertainty principles says that $\Delta x\Delta p > \frac{h}{4\pi} = 5.27\times10^{-35}\,\mathrm{J\,s}$.[2] We know that Δx is $0.5\,\mu$m since that is how accurately we can tell where the particle is at any given time and, in particular, at the moment at which we verify whether or not it has crossed one of the rulings. $\Delta p = m\Delta v$ so we need to calculate the mass of the particle: The diameter is $1\,\mu$m so the radius is $0.5\,\mu$m. The volume of the spore is

   $$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(0.5\times10^{-6}\,\mathrm{m})^3 = 5.2\times10^{-19}\,\mathrm{m}^3 \equiv 5.2\times10^{-13}\,\mathrm{cm}^3.$$

   Using the density, we find that the spore weighs approximately $5.2\times10^{-13}\,\mathrm{g}$ or $5.2\times10^{-16}\,\mathrm{kg}$. The uncertainty in p is therefore

   $$\Delta p = (5.2\times10^{-16}\,\mathrm{kg})(4\times10^{-9}\,\mathrm{m/s}) = 2.1\times10^{-24}\,\mathrm{kg\,m/s}.$$

   Thus, $\Delta x\Delta p = 1.0\times10^{-30}\,\mathrm{J\,s}$, which is about 20 000 times larger than the lower limit imposed by the Heisenberg uncertainty principle. There is therefore plenty of room to improve this experiment.

2. Since $E = hc/\lambda$, the longest-wavelength transitions correspond to the smallest differences in energy. The energy levels of a particle in a box go like n^2, so the smallest energy gap in the spectrum is between the $n = 1$ and $n = 2$ levels. This has a ΔE of

   $$\Delta E_1 = \frac{hc}{\lambda_1} = \frac{2^2h^2}{8mL^2} - \frac{1^2h^2}{8mL^2} = \frac{3h^2}{8mL^2}.$$

   We can use this relationship to obtain an initial estimate of L:

   $$L^2 \approx \frac{3h\lambda_1}{8mc}$$

[1] A more sophisticated treatment of experimental uncertainty would make the net uncertainty in the distance travelled $\sqrt{2}(0.5\,\mu\mathrm{m}) = 0.7\,\mu$m.

[2] I normally write the units of h as J/Hz. However, one interpretation of the factor of 2π dividing $h/2$ in the Heisenberg relation is that it converts Hz (cycles/s) to s^{-1} (radians/s).

$$
\begin{aligned}
&= \frac{3(6.626\,068\,8\times10^{-34}\,\mathrm{J/Hz})(1104\times10^{-9}\,\mathrm{m})}{8(9.109\,381\,9\times10^{-31}\,\mathrm{kg})(2.997\,924\,58\times10^{8}\,\mathrm{m/s})} \\
&= 1.004\times10^{-18}\,\mathrm{m}^2. \\
\therefore L &\approx 1.0022\times10^{-9}\,\mathrm{m}.
\end{aligned}
$$

There is of course some experimental error in all measurements, so we must identify the other lines and use these points as well in obtaining our final estimate. We do this by trial and error, we know that for each observation j,

$$\Delta E_j = \frac{hc}{\lambda_j} = \frac{h^2}{8mL^2}\left(n_{f,j}^2 - n_{i,j}^2\right)$$

where $n_{f,j}$ and $n_{i,j}$ are the final and initial values of the quantum number. Therefore

$$n_{f,j}^2 - n_{i,j}^2 = \frac{8mcL^2}{h\lambda_j}.$$

We then just need to find two integers which satisfy this relationship, at least approximately. Having done that, we can calculate L from each data point by rearranging the equation to

$$L^2 = \frac{h\lambda_j}{8mc}\left(n_{f,j}^2 - n_{i,j}^2\right).$$

My results are as follows:

λ_j (nm)	$n_{f,j}^2 - n_{i,j}^2$	$n_{f,j}$	$n_{i,j}$	L (nm)
663	5	3	2	1.0027
473	7	4	3	1.0021
414	8	3	1	1.0022

The box length is obtained by averaging the four measurements. This gives $L = 1.0023\,\mathrm{nm}$.

3. According to the Bohr theory, the orbital speed of the electron is

$$v^2 = \frac{Ze^2}{4\pi\varepsilon_0 m_e r}.$$

The orbital radius is fixed by the quantization condition:

$$r = \frac{n^2h^2\varepsilon_0}{\pi Ze^2 m_e}.$$

Accordingly, the speed is related to n by

$$v_n^2 = \frac{Z^2e^4}{4n^2h^2\varepsilon_0^2}.$$

In our case, $Z = 92$. The largest speed (which will cause the most serious problems for a nonrelativistic theory) is obtained when $n = 1$.

$$\begin{aligned} v_1^2 &= \frac{(92^2)(1.602\,176\,46 \times 10^{-19}\,\mathrm{C})^4}{4(1^2)(6.626\,068\,8 \times 10^{-34}\,\mathrm{J/Hz})^2(8.854\,187\,817 \times 10^{-12}\,\mathrm{C^2J^{-1}m^{-1}})^2} \\ &= 4.05 \times 10^{16}\,\mathrm{m^2/s^2}. \\ \therefore v_1 &= 2.01 \times 10^8\,\mathrm{m/s}. \end{aligned}$$

This is about $\frac{2}{3}$ of the speed of light. At such high speeds, relativistic corrections are always needed.

4. Some possible answers:

- The uncertainty principle rules out the possibility of orbits of fixed radius because the radial position is then exactly known which means that the uncertainty in the component of the momentum transverse to the orbit is infinite.
- The Bohr theory predicts an angular momentum vector of size $|\mathbf{L}| = rp = rmv$ which depends on the quantum number n (because r and v depend on n). In fact, the orbital angular momentum depends on a completely different quantum number ℓ.