THOUGHTS ON TEACHING ELEMENTARY MATHEMATICS*

Jonathan P. Seldin
Department of Mathematics and Computer Science
University of Lethbridge
Lethbridge, Alberta, Canada
jonathan.seldin@uleth.ca
http://www.cs.uleth.ca/∼seldin

October 27, 2006

Keith Devlin [3] made a distinction between *elementary mathematics* and *formal mathematics*. His idea was that the latter is characterized by definitions which one must use in order to come to understand, whereas the in the former, each new concept follows from previously understood concepts in a natural way and can be understood directly. Devlin suggested that formal mathematics starts with calculus, or, more specifically, with the limit concept,[1] and he also suggested that many people may not be able to learn formal mathematics.

It seems to me that Devlin's observation is not a distinction between kinds of mathematics but a distinction between different ways mathematics is communicated. After all, new definitions created by research mathematicians do not occur to them in the formal manner described by Devlin, but follow from the experience of research the way Devlin characterizes elementary mathematics. It is only when the research results are written up for publication that the definitions are introduced in this formal way.

This idea has implications for the teaching of mathematics. Virtually all departments of mathematics have courses designed to help students make the transition

*This work was supported in part by a grant from the Natural Sciences and Engineering Research Council of Canada.

[1]Or, more exactly, with the formal ϵ-δ definition of a limit.

from elementary courses (through the calculus sequence) to advanced courses (which require proofs). Clearly, mathematics majors need to be able, in their advanced courses, to learn mathematics that is presented formally. Devlin's distinction suggests strongly that the material of this transition should not be presented formally. But most textbooks I have seen designed for this transition course[2] start with formal logic presented formally. I think Devlin's talk implies strongly that the material of such a transition course should *not* be presented formally, and neither should the mathematics presented in any of the elementary courses which precede this transition course. Instead, it should be presented in such a way that each new concept follows directly from what has been previously learned, and the formal definitions involved are only presented as a kind of summary after the ideas themselves have been mastered. I think that presenting elementary mathematics this way will amount to retracing some of the high points of the process by which modern mathematics developed.

The purpose of this paper is to suggest some ways in which formal presentation can be avoided in the elementary part of the mathematics curriculum and to suggest how students can be helped to make the transition to the learning of formally presented mathematics. I intend to start where the problems most students seem to have with formally presented mathematics begins: with the teaching of elementary algebra.

I would like to thank Roger Hindley for his helpful comments and suggestions.

1 Elementary Algebra

Many students that I see in college and university level mathematics courses seem to believe that algebra is a game involving the formal manipulation of symbols that have, for them, little or no meaning. These students seem to have weak algebraic skills, and this causes them problems throughout their mathematical studies. Among these problems are a major difficulty solving word problems. For those who go on to study computer science, it shows up in students who learn to write code easily but have trouble solving problems.

One example of the difficulties some students have in writing equations for word problems is given in [5, pp. 65–66], where there is a discussion of the tendency to express the fact that there are six times as many students as professors by $6s = p$ instead of $6p = s$. Is this an example of difficulties that arise because algebra is

[2] Including those at the book exhibit at the meeting at which Devlin presented [3] and Devlin's own book [4].

presented in a formal way?

I think one of the main differences between arithmetic and algebra from the point of view of students is that whereas an arithmetic problem tells the student which mathematical operations to perform, in solving an algebraic equation one has to determine which operations to perform by indirect reasoning. The other is that algebra is also concerned with general properties of numbers, properties which must be determined by reasoning.

Thus, I maintain that the key to algebra is *reasoning.*

This idea is not original with me: see the preface of [7]. Here, Middlemiss makes the point that in solving the equation $3x+5=26$ by subtracting 5 from both sides to get $3x=21$ and then dividing by 3 to get $x=7$, one is proving that: *if* $3x+5=26$, *then* $x=7$. Checking the solution by substituting 7 for x in the left side of the original equation and carrying out the calculations, getting $3(7)+5=21+5=26$ proves the converse, *if* $x=7$, *then* $3x+5=26$. And that is a different matter from solving the original equation.

Middlemiss goes on to point out that students who do not understand this may be lost in trying to solve

$$\sqrt{4x+5}=2\sqrt{x+1}.$$

Squaring both sides leads to

$$4x+5=4(x+1)$$

and then to

$$5=4.$$

Students who think algebra is a game involving manipulating symbols may be lost here. But students who understand the reasoning involved will realize that this is a proof of *if* $\sqrt{4x+5}=2\sqrt{x+1}$, *then* $5=4$, and this, in turn, proves that *the equation* $\sqrt{4x+5}=2\sqrt{x+1}$ *is false for all values of* x.

I maintain that understanding the reasoning involved here requires us to understand the meaning of the symbols. Furthermore, in simple cases, this can be done without the symbols, which is the way all algebraic problems were solved before our modern algebraic notation developed out of a kind of shorthand. As an example, consider the problem: find a number such that five more than three times the number is twenty. This problem can be solved in words as follows:[3]

Five more than three times the number is twenty.
Three times the number is fifteen.
The number is five.

[3] I first saw this example in a seminar run by N. Herscovics on the teaching of secondary school mathematics at Concordia University, Montreal, fall 1982.

This proves: *If five more than three times the number is twenty, then the number is five.*

The process of checking the solution is as follows: Five more than three times five is five more than fifteen, which is twenty. This proves: *if the number is five, then five more than three times the number is twenty.*

Note that solving simple equations without the symbols is a non-formal presentation that emphasizes the importance of the reasoning. My proposal is that a beginning algebra course should start with this kind of non-formal introduction, and should only introduce the algebraic notation at a later stage. Perhaps after this non-formal presentation of solving simple algebraic problems could be followed by a presentation of carefully selected logical puzzles (in words) to help the students learn to reason about such matters. Furthermore, the introduction of the algebraic notation might be postponed until the students start complaining about the amount of writing they have to do to solve the problems. And if there is time, the students might first be encouraged to introduce their own shorthand for this kind of solution and only introduced to the standard notation after they have spent some time working on the difficulties of find a shorthand that presents all the necessary information. (Of course, some students may come up with the standard notation themselves before that happens.)

In proposing this, I am not denying the view expressed by Devlin in [2] that learning mathematics requires students to learn to follow mechanical rules without understanding them and gaining facility by practice. There are many patterns of reasoning used in elementary algebra that many students will need to learn this way. But this does not require that students start learning these rules with the algebraic notation. It is this latter practice which I think is at the root of most of the confusion that students have about algebra. True, this algebraic notation represents the language used throughout all subsequent mathematics that students will learn. But this does not mean that this language should be introduced with the first algebraic algorithms that the students learn. Eventually they will all need to learn it, but just as nobody tries to teach the grammar of their mother tongue to students who have not learned to speak, read, and write it, so I think the notation of algebra should not be taught to students who have never done any algebra at all.

2 Mathematical Reasoning

If the essence of all mathematics beyond elementary algebra is reasoning, mathematical reasoning differs in some important respects from ordinary reasoning in everyday life. In ordinary reasoning, we accept generalizations to be true even if they have exceptions. For example, we accept as a true generalization that leopards are four-legged animals, even though there may be leopards with fewer than four legs because of birth defects.[4] But in mathematics, a universal statement is not accepted as true if there are any exceptions. Or, to take another example, in mathematics, we insist on proofs of results even if they are intuitively obvious, but this is not the case with legal reasoning¿ magine a judge's reaction to a lawyer saying in a trial that he needs to use the notion of equality in his argument and that therefore he needs to prove that everything is equal to itself!

There was a time when students learned what a mathematical proof is (and hence what mathematical reasoning is like) from a course in plain geometry whose proofs were in the style of Euclid's *Elements*, but at least in North America these courses have disappeared from the curriculum, and there are not enough teachers competent to teach such a course to re-introduce it easily.

Fortunately, it is possible without such a course to use episodes from the history of mathematics to illuminate the notion of mathematical proof.

In fact, it may be useful to go back to the period of mathematics before Euclid to discuss how the notion of proof found in the *Elements* developed.[5] The kind of arguments that we now often associate with formal debates were, in a less formal way, a regular part of daily life. Furthermore, a man who wanted to become an advocate did not study the law (the statutes and precedents) as he would today, but instead studied rhetoric, or how to argue effectively. There developed a class of teachers known as sophists who would teach students (for a fee) how to speak persuasively on many different kinds of topics. For an example of the kind of argument that occurred in this society, see the story of Protagoras and Euathlus in [1, p. 10].[6]

It is thought that the earliest proofs given by the ancient Greeks were not sequences of statements, but rather a matter of looking at diagrams, seeing the result in question for that case, and realizing that a diagram for any case covered by the proof would give rise to a similar diagram from which the result would be apparent.[7] I remember learning formulas for the areas of some geometric figures this way. Once I knew that the area of a rectangle was the base times the height, I learned the formula for a parallelogram from Figure 1.
Similarly, I learned the formula for the area of a triangle from Figure 2
It is also possible to see the proof of the Pythagorean Theorem from Figure 3.
Also, to note the use of diagrams in ancient texts, note that Plato, in *Meno*, used

[4] I am indebted to John Woods for this example.

[5] This material is from [8].

[6] The story can also be found in [8].

[7] See [13, pp.2–3] and [10, pp. 258–259].

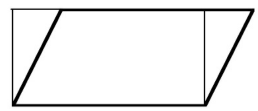

Figure 1: Area of Parallelogram

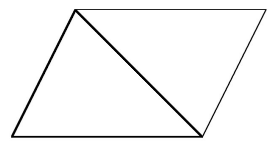

Figure 2: Area of Triangle

Figure 4 to show how to double the area of a square.

An explanation of the change in proofs from this kind of diagram proof to proofs as sequences of statements is given by Knorr in [6, Chapters V–VI] and is also outlined in my paper [8]. This presentation of the incommensurabitiy of the side and diagonal of a square seems less formal than the traditional proof, which assumes that $\sqrt{2}$ is equal to $\frac{p}{q}$ where p and q are relatively prime and derives a contradiction from that.

It also provides a chance to discuss what it might mean for a unit to divide *exactly* into the length of a line segment. The idea indicates the kind of idealization that is a part of mathematics.

3 Calculus and Beyond

The subject that Devlin emphasized in his talk was calculus: he presented it as the first formal mathematics.

It seems to me that calculus can be presented without that formal definition. Engineering students often learn to use calculus without studying that formal definition, and surely other students can as well.[8] The *theory* of the calculus can be left out of the first course and presented later, in a third-year analysis course. In my view, this analysis course should be presented non-formally, as a critique of the calculus. And it should follow a non-formal transition course. I am not aware of any

[8]One book that presents calculus in this way is [11], but that book is missing many of the topics now included in a standard course on calculus, and the chapters Martin Gardner added to the new edition [12] appear to move in the direction of a formal presentation.

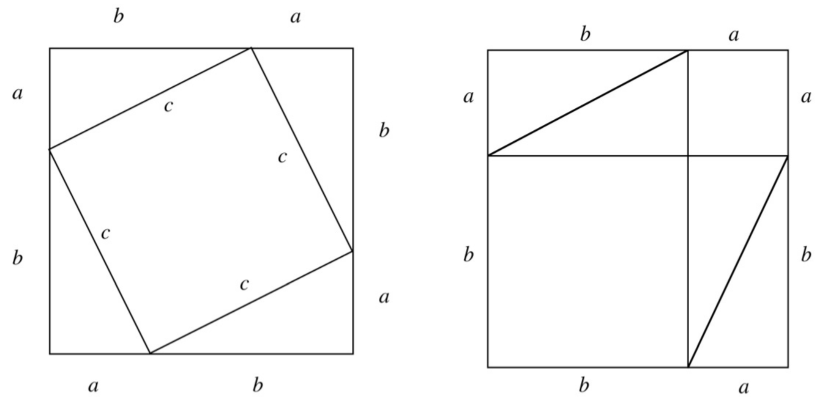

Figure 3: Pythagorean Theorem

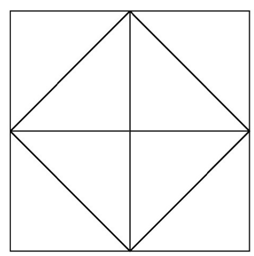

Figure 4: Doubling a square

textbooks for either of these courses that present them non-formally. What follows are suggested outlines for each of these courses.

Example 1 Proposed outline for a transition course Catalog description: Logic, proofs. Set theory. Relations and functions. Finite and countable sets. Induction. Examples of axiomatic mathematical theories.

Proposal for course outline:

1. Basic intro to math reasoning. Ideas from [8]. Start with solution of simple algebraic equations. Cover change from reasoning about diagrams to proofs as sequences of statements. Include proof of irrationality of square root of 2 as done in [6] and in [8]. Discuss why this shows the nature of idealization in mathematics; i.e., what it would mean to have a unit that measures *exactly* the length of a line segment. Perhaps discuss here the difference between the ancient Greek approach to number

and magnitude and the modern approach (which apparently originated in the Arab world during the middle ages and spread to Europe during the Renaissance).

2. Introduce equivalence relations by starting with fractions and rational numbers. Then show that the same procedure can lead from positive integers (rationals, reals) to all integers (rationals, reals) by using formal differences. Emphasize that the same pattern occurs in both cases, and generalize to equivalence relations and classes. In both cases, show the need to prove that the definitions of operations respect the equivalence relation. (Discuss complex numbers, where equivalence relation is identity?)

3. As a special case of equivalence relations, discuss cardinal numbers. Start writing results in words rather than symbols, and show countability of perfect squares, odds, evens, etc. Discuss nature of infinite sets. Prove uncountability of reals. Discuss functions, using one-to-one correspondences to introduce properties of functions. Show set of real valued functions of a real variable does not have cardinality of continuum. Use this material to discuss the nature of sets, and slowly introduce language of set theory for this discussion. Discuss the difference between containment and membership (which Dedekind ignored) and singleton sets. Discuss the empty sets. (Discuss coding functions and relations as their graphs (sets of ordered pairs)? Introduce ideas behind lambda-calculus (functions as rules of calculation) for computer science students?)

4. Discuss mathematical induction. Generalize to inductive constructions and induction in general. Discuss inductive definitions of addition and multiplication in connection with the Peano postulates, and indicate why this can be an axiomatic basis for natural numbers. (Discuss difference between including and excluding 1?)

5. Critically examine proofs given so far. Discuss NOT, AND, and OR in Excel. Discuss truth tables. Introduce symbols for not, and, and or. Discuss material implication and truth tables in general. Discuss use of propositional calculus to evaluate arguments.

6. Perhaps include discussion of some axiomatic theories in a form that allows instructor to choose. Possibilities include

A. Group theory. Start with integers under + and positive rationals under times, and generalize.

B. Ring theory. Start with integers, rationals, and reals under + and times. Generalize from examples.

C. Field theory. Start with rationals and reals under + and times, and generalize.

D. Elementary geometry. Start with Euclid's axioms, most of which can be illustrated with simple diagrams. Show that these axioms and postulates seem to be satisfied by the sets of ordered pairs (or triples) with rational coefficients, and show the problem with applying Prop. 1 of Book I to the unit line segment from the origin on the x-axis: the third vertex of the equilateral triangle has an irrational coefficient.

In cases A, B, and C, start with examples and develop the axiomatic theory by looking for common patterns. Since geometry can be more easily visualized than the previous examples, the approach can be different here.

Example 2 Proposed Outline for Third-year analysis course

Catalog description: Rigorous treatment of the notions of calculus of a single variable, emphasizing epsilon-delta proofs. Completeness of the real numbers. Upper and lower limits. Continuity. Differentiability. Riemann integrability.

Proposed course outline:

1. Intro to critique of calculus. In evaluation of derivative of $f(x)$ at $x = a$, we do manipulations that depend on x not being a and then substitute a for x. Why can we do this? Berkeley's "The Analyst". How can we justify using calculus? What constitutes a proof in mathematics? (Ideas from outline for transition course if students have not had it.)

2. Ancient Greek approach to numbers and magnitudes. Zeno's paradoxes. Some pre-Euclidean proofs. Knorr's proof that the side and diagonal of a square are incommensurable. Discussion of this and what it could mean to measure a length *exactly*.

3. Idea behind limits. Preliminary theory of limits. Have students list limit theorems from their calculus text (or give them such a list) and have them determine which ones need to be taken as axioms from which the others can be proved. Show that this is incomplete. Then use [9] to get $\epsilon - N$ definition, and from this get $\epsilon - \delta$.

4. What must be true of quantities or numbers for all this to work? Why rationals are not enough. How can we get reals? Evolution from ancient Greek ideas to modern ideas. Construction of number systems.

5. Countable and uncountable sets. Start with Galileo's observation that positive integers can be paired with the positive perfect squares. Additional examples. Uncountability of reals, and proof that set of real valued functions of real numbers has higher cardinality than reals. Introduce language of set theory from this, and also discuss functions and their properties.

6. Metric topological properties of real numbers. More on limits. Continuity. Sequences, series, and convergence.

7. Derivatives. Use of limits in defining. The derivative is a function of a function.

8. Riemann integrability. Properties of lubs and glbs needed to prove properties.

9. If time, measure theory and Lebesgue integral. Start with question of what sets of points of discontinuity a function can have and still have an integral, based on Fourier series.

References

[1] W. DeLong. *A Profile of Mathematical Logic*. Addison-Wesley, Reading, MA & Menlo Park, CA & London & Don Mills, Ontario, 1970.

[2] K. Devlin. How do we learn math? Devlin's Angle column, March 2006, MAA website, http://www.maa.org/devlin/devlin_03_06.html.

[3] K. Devlin. How much mathematics can be for all? Plenary Talk, Summer 2005 Meeting of Canadian Mathematical Society with Canadian Society for History and Philosophy of Mathematics, University of Waterloo, Waterloo, Ontario, June 4–6, 2005.

[4] K. Devlin. *Sets, Functions, and Logic: An Introduction to Abstract Mathematics*. Chapman & Hall/CRC, Boca Raton, LA & London & New York & Washington, DC, third edition, 2004.

[5] N. Herscovics. Cognitive obstacles encountered in the learning of algebra. In S. Wagner and C. Kieran, editors, *Research Issues in the Learning and Teaching of Algebra*, volume 4, pages 60–86. Lawrence Erlbaum Associates, National Council of Teachers of Mathematics, 1989.

[6] W. R. Knorr. *The Evolution of the Euclidean Elements: A Study of the Theory of Incommensurable Magnitudes and Its Significance for Early Greek Geometry*. Reidel, Dordrecht and Boston and London, 1975.

[7] R. R. Middlemiss. *College Algebra*. McGraw-Hill, 1952.

[8] J. P. Seldin. Reasoning in elementary mathematics. In Tessoula Berggren, editor, *Canadian Society for History and Philosophy of Mathematics. Proceedings of the Fifteenth Annual Meeting, Quebec City, Quebec, May 29–May 30, 1989*, pages 151–174, 1989. Available on line from `http://www.cs.uleth.ca/~seldin` under "Publications.".

[9] J. P. Seldin. From exhaustion to modern limit theory. In Francine F. Abeles, Victor J. Katz, and Robert S. D. Thomas, editors, *Canadian Society for History and Philosophy of Mathematics. Proceedings of the Sixteenth Annual Meeting, Victoria, British Columbia, May 31–June 1, 1990*, pages 120–136, 1990. Available on line from `http://www.cs.uleth.ca/~seldin` under "Publications.".

[10] E. Stenius. Foundations of mathematics: Ancient greek and modern. *Dielectica*, 32(3–4):255–290, 1978.

[11] Silvanus P. Thompson. *Calculus Made Easy: Being a Very-Simplest Introduction to Those Beautiful Methods of Reckoning which are Generally Called by the Terrifying Names of the Differential Calculus and the Integral Calculus*. Macmillan, third edition, 1946.

[12] Silvanus P. Thompson and Martin Gardner. *Calculus Made Easy: Being a Very-Simplest Introduction to Those Beautiful Methods of Reckoning which are Generally Called by the Terrifying Names of the Differential Calculus and the Integral Calculus*. St. Martin's Press, New York, 1998. Newly Revised, Updated, Expanded, and Annotated for its 1998 edition.

[13] R. Toretti. *Philosophy of Geometry from Riemann to Poincaré*. Reidel, Dordrecht: Holland/Boston: U.S.A./London:England, 1978.