Gödel, Kuhn, and Feyerabend*

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Abstract

Thomas Kuhn [4] has presented a view of the history of science as a succession of “paradigms” which are not completely comparable with each other. Karl Popper [8] has attacked this view as being relativistic and denying that there is objective scientific truth. Popper seems to be saying that in order for science to be objectively true, every two scientific theories must be completely comparable. In taking this position, Popper is making a claim for a kind of completeness that is ruled out for theories strong enough to be “interesting” by Gödel’s Incompleteness Theorem. In this paper, Gödel’s Incompleteness Theorem and its proof will be used to draw conclusions about this conflict between Kuhn and Popper. In particular, it will be argued that if Popper had taken Gödel’s Incompleteness Theorem into account, he would have wound up with a position consistent with that of Kuhn. Gödel’s Incompleteness Theorem will also be used to argue that Feyerabend [3] does not really have an argument that there is no objective scientific method. The paper will close with some remarks on what might really count as objective scientific truth.

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1 The dispute between Kuhn and Popper

Thomas Kuhn [4] has presented a view of the history of science that differs from the previously commonly held view of science as a gradual accumulation of knowledge.¹ He presents the history of science as a succession of periods of “normal science,” each of which is determined by a “paradigm.” When for some reason the paradigm is no longer adequate for normal science to proceed in the usual way, there is a “scientific revolution,” and normal science only resumes when a new paradigm is accepted by the scientific community involved.²

One of the most controversial aspects of Kuhn’s ideas is his view that successive paradigms are not completely compatible. He says, for example, “Like the choice between competing political institutions, that between competing paradigms proves to be a choice between incompatible modes of community life.”³ Later, he says, “In so far as their only recourse to that world is through what they see and do, we may want to say that after a revolution scientists are responding to a different world.”⁴

Karl Popper objected to this view on the grounds that it was relativism.⁵ In [8, p. 56], Popper says of Kuhn’s views:

...Kuhn suggests that the rationality of science presupposes the acceptance of a common framework. He suggests that rationality depends upon something like a common language and a common set of assumptions. He suggests that rational discussion, and rational criticism, is only possible if we have agreed on the fundamentals.

This is a widely accepted and indeed a fashionable thesis: the thesis of relativism. And it is a logical thesis.

¹Popper’s view of science also differed from that previously commonly held view. In [7], he proposed the view that science consists of the bold formation of theories as hypotheses which are then subjected to attempts to falsify them.
²Usually, this is a group of specialists, but sometimes it may be a larger group than that.
⁴Ibid., p. 111.
⁵Popper was not the only one who thought that Kuhn was relativist: “There is nobody else than Thomas Kuhn who contributed more to the widespread acceptance of cognitive relativism in the recent years.” [14, p. 25].
I regard this thesis as mistaken. ...\(^6\)

And he goes on to say later on the same page:

I should like just to indicate briefly why I am not a relativist:\(^7\) I do believe in ‘absolute’ or ‘objective’ truth, in Tarski’s sense (although I am, of course, not an ‘absolutist’ in the sense of thinking that I, or anybody else, has the truth in his pocket.) I do not doubt that this is one of the points on which we are most deeply divided; and it is a logical point.

And again later on the same page:

The central point is that a critical discussion and a comparison of the various frameworks is always possible. It is just a dogma—a dangerous dogma—that the different frameworks are like mutually untranslatable languages.

Popper did eventually accept that he had misinterpreted Kuhn’s views. He says of the view that comparison of different scientific theories requires an agreement on the general framework, a view with which he disagrees:

... I originally had in mind Thomas Kuhn .... ... However, as Kuhn points out, this interpretation was based on a misunderstanding of his views (see his [5], and his ‘Postscript 1969’ to [4]), and I am very ready to accept his correction. Nevertheless, I regard the view here discussed as influential.\(^8\)

There seem to be other disagreements between Kuhn and Popper. Kuhn concludes that it may not be possible to say that as science progresses it is bringing scientists closer to the truth:

... scientific progress is not quite what we had taken it to be. ... In the sciences there need not be progress of another sort. We may, to be more precise, have to relinquish the notion, explicit or implicit, that changes of paradigm carry scientists and those who learn from them closer and closer to the truth.\(^9\)

\(^6\)The emphasis is in the original.

\(^7\)Author’s footnote: See, for example, Chapter 10 of my *Conjectures and Refutations*, and the first *Addendum* to the 4th (1962) and later editions of volume ii of my *Open Society*.

\(^8\)See [13, Footnote 19, p. 63].

\(^9\)[4, p. 170].
On the other hand, Popper says about scientific theories: “The aim is to find theories which, in the light of critical discussion, get nearer to the truth.”

An indication of what Popper means by this is as follows:

I shall give here a somewhat unsystematic list of six types of case in which we should be inclined to say of a theory $t_1$ that it is superseded by $t_2$ in the sense that $t_2$ seems—as far as we know—to correspond better to the facts than $t_1$ in some sense or other.

(1) $t_2$ makes more precise assertions than $t_1$, and these more precise assertions stand up to more precise tests.

(2) $t_2$ takes account of, and explains, more facts than $t_1$ (which will include, for example the above case that, other things being equal, $t_2$’s assertions are more precise).

(3) $t_2$ describes, or explains, the facts in more detail than $t_1$.

(4) $t_2$ has passed tests which $t_1$ has failed to pass.

(5) $t_2$ has suggested new experimental tests, not considered before $t_2$ was designed (and not suggested by $t_1$, and perhaps not even applicable to $t_1$); and $t_2$ has passed these tests.

(6) $t_2$ has unified or connected various hitherto unrelated problems.

If we reflect upon this list, then we can see that the contents of the theories $t_1$ and $t_2$ play an important role in it. (It will be remembered that the logical content of a statement or theory $a$ is the class of all statements which follow logically from $a$, while I have defined the empirical content of $a$ as the class of all basic statements which contradict $a$.11) For in our list of six cases, the empirical content of theory $t_2$ exceeds that of theory $t_1$.12

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10 [8, p. 57]

11 Author’s footnote: This definition is logically justified by the theorem that, so far as the ‘empirical part’ of the logical content is concerned, comparison of empirical contents and of logical contents always yield the same results; and it is intuitively justified by the consideration that a statement $a$ tells the more about our world of experience the more possible experiences it excludes (or forbids). . .

12 [11, p. 232].
This suggests that Popper is aiming to give a set of deductive rules to express the idea that one theory is nearer to the truth than another. The idea presupposes the kind of relationship between theories that Kuhn claims does not always exist between paradigms, but is excluded by the incommensurability between paradigms, especially if by incommensurability one means the notion of local incommensurability of [6].

I believe that if Popper had taken more account of Gödel’s Incompleteness Theorem, he would have found his position much closer to that of Kuhn than he realized.\footnote{Popper was certainly aware of Gödel’s Incompleteness Theorem; he refers to it, for example, in [9, pp. 269ff] and [10, p. 310]. But he seems to have missed this particular application of it.}

2 Gödel’s Incompleteness Theorem and the Kuhn-Popper Dispute

For a discussion of Gödel’s Incompleteness Theorem and its proof, see Appendix B.

To see how this theorem might apply to the dispute between Popper and Kuhn, suppose that $T_1$ and $T_2$ are two scientific theories to be compared. For example, $T_2$ might be a theory which has replaced $T_1$ in what Kuhn calls a scientific revolution. Any comparison would have to be carried out in some third theory, $T'$. The theory $T'$ will have to be strong enough to include references to $T_1$ and $T_2$ and to draw conclusions from those references. Assuming that $T_1$ and $T_2$ are “strong enough to be interesting,” $T'$ will almost certainly have to be strong enough to represent all partial recursive functions. This means that if $T'$ is completely formalized, Gödel’s Incompleteness Theorem will apply to $T'$. It follows that there cannot be a guarantee that it will be possible in $T'$ to obtain a complete comparison of all aspects of $T_1$ and $T_2$. This is why I claim that Kuhn’s notion of incommensurability, especially in the version of local incommensurability of [6], should be considered an expected consequence of Gödel’s Incompleteness Theorem.

A similar argument will, I believe, show why we cannot talk about science progressing to theories that come ever closer to the truth about the world. To say that one theory, say $T_1$, is closer to the truth than another, say $T_2$, requires still another theory, $T'$, in which to carry out this reasoning. Since we are
dealing with scientific theories, it is reasonable to suppose that $T'$ would, if completely formalized, be subject to Gödel’s Incompleteness Theorem. This would mean that there would be truths about closeness to the truth which would be unprovable in the formalization of $T'$, and this implies that it is not always possible to say that newer theories are closer to the truth than older ones. Thus, we should consider it a consequence of Gödel’s Incompleteness Theorem that we cannot always talk about scientific progress getting closer to the truth.

Of course, most scientific theories are not presented in a completely formalized form, and Gödel’s Incompleteness Theorem only applies directly to strictly formalized theories. But the process of formalization is mainly a matter of making definitions more precise and bringing out hidden assumptions until it is possible to decide in an essentially mechanical way whether or not any given sequence of sentences constitutes a valid deduction. Most mathematical proofs are presented in a way which suggests that they could be carried out in a completely formal system if somebody were willing to do the necessary work, and to the extent that scientific theories are deductive and use mathematics, they are like pure mathematical proofs in this regard. I think that this implies that the scientific theories we are considering could be formalized, and thus that Gödel’s Incompleteness Theorem applies to them.

It thus appears that Popper was originally more mistaken about the Kuhn’s views being relativist and subjective than he later realized. This seems especially likely since Kuhn implies in [4] that every paradigm will eventually be found inadequate: surely this implies that there is some sort of objective reality against which these paradigms are being tested. The implication of the incommensurability of paradigms\footnote{Kuhn’s claim of incommensurability of paradigms is based on an analysis of the history of science rather than on any logical analysis.} is that our ability to know and talk about this objective reality is limited, not that this objective reality does not exist. In fact, Kuhn himself [5] points out that except for this one difference,\footnote{I.e., Popper’s claim that the incomparability of paradigms implies relativity.} his views are very similar to those of Karl Popper, and when he and Popper are analyzing the same historical episode in science their analyses are usually very much alike. Perhaps we can say that, roughly,

$$\text{Popper + Gödel = Kuhn.}$$
3 Gödel’s Incompleteness Theorem and Paul Feyerabend

Gödel’s Incompleteness Theorem also has something to tell us about the views expressed by Paul Feyerabend in [3].16 Feyerabend thinks that he is arguing that there is no such thing as a scientific method. But when his arguments are stripped of rhetoric, what is left is a catalog of instances in which various descriptions of scientific methodology fail to correspond with the way science has actually progressed. Feyerabend argues correctly that this shows that the descriptions of scientific methodology of which he is speaking do not fit science as it has actually occurred, and he goes on to suggest that no complete and consistent description of scientific methodology is possible.

Feyerabend then argues that the absence of a sound and complete description of the scientific method means that there is no such method. But, again, this argument is based on the same kind of claim of completeness as Popper’s. In fact, Gödel’s Incompleteness Theorem should lead us to expect that no description of the scientific method can be both sound and complete. Feyerabend may have produced some evidence that we can never soundly and completely describe the scientific method, but he has no evidence whatever that there is no such method.17 The evidence Feyerabend presents in [3] simply does not support his thesis that “anything goes” in science.

4 The objective truth of science

At this point some readers may wonder what we can say to those who really deny the value or truth of science. I think there is an answer to them, but it is not the same kind of answer that Popper was seeking. The answer I have in mind does not try to prove the truth of every assertion of the accepted scientific paradigms. Instead, it concentrates on those questions that are of interest to legislative bodies and courts involving matters of public policy or the value of scientific testimony. Legislative bodies and courts are interested

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16 This is the only work by Feyerabend that I have read. My remarks in this section apply only to [3], and not to any of his other works.

17 If there can be no sound and complete description of a scientific method, perhaps “method” is the wrong word: perhaps we should be speaking of a scientific approach.
in evidence for their conclusions on these issues. And I think that the refereeing procedure that is standard in the best scientific journals is the procedure that generates the best evidence for or against the assertions involved.\footnote{Ayala and Black \cite{1} argue from the position of Popper that the courts should pay attention to refereed scientific journals.} This gives us a basis for saying that there is an objective basis for these assertions. Of course, at a later time, the evidence may change with regard to any particular question, but \textit{at the time the question is being considered}, a consensus of the literature by specialists on the matter in question in refereed scientific journals, if such a consensus exists, is the best evidence available on that question.\footnote{It is possible that at a given time there is no consensus of the literature by specialists on the matter in refereed scientific journals. In this case, the legislatures and courts would have to conclude that there is insufficient scientific evidence to settle the matter at hand scientifically.} This is the way we can refer to science as embodying objective truth.

There is one important qualification we must make here: it is important that this refereeing process be conducted in a way consistent with the practices of the scientific speciality in question and is free from external influences, such as economic interests. I think most of us know from our own personal experience that if we have a reason to want something to be the case, it is easy to convince ourselves that it is the case even if the evidence does not support that conclusion. This means that each of us must constantly be on guard against making this kind of mistake in our own individual work. Economic interest can cause a very powerful desire for something to be the case even if it is not. This suggests that in areas of science in which there are important economic interests that are affected by the results of scientific research, special efforts are needed to protect the process of scientific research and refereeing from the effects of those economic interests.\footnote{For a troubling example showing how medical research financed by a pharmaceutical company may be affected by economic interests, see \cite{2}.}

Of course, there are questions that arise in science that are never considered by courts or legislative bodies: an example is the question that arose in the early days of the special theory of relativity, when there was an alternative theory due to H. A. Lorenz based on classical physics which made exactly the same predictions as Einstein’s theory. Einstein’s arguments for his approach did not constitute the same kind of evidence from the point of view of courts and legislative bodies as, say, the scientific evidence that AIDS
is caused by the HIV virus. But then these arguments of Einstein were on a subject unlikely to get before a court or legislative body. It is at the point at which science touches public policy or trials through courts and legislative bodies that we are best able to talk about the objective reality of science.  

Appendices

A A formal system for first-order arithmetic

This is a specification of classical first-order arithmetic. It is an example of the kind of system to which Gödel’s Incompleteness Theorem applies.

Definition 1 We assume that we are given infinitely many variables: $x_0, x_1, x_2, \ldots$. We will denote these variables by $x, y, z, \text{etc.}$ Terms are then defined as follows:

1. Every variable is a term.
2. The constant 0 is a term.
3. If $s$ is a term, so is $s'$.
4. If $s$ and $t$ are terms, so are $s + t$ and $st$.

We will use $'m^*$' as an abbreviation for $0 \overset{t \cdots t}{\ldots m}$, the term representing the number $m$.

Definition 2 Formulas are defined as follows:

1. If $s$ and $t$ are terms, then $s = t$ is a formula.
2. If $A$ is a formula, then so is $\neg A$.
3. If $A$ and $B$ are formulas, then so is $A \supset B$.
4. If $A$ is a formula and $x$ is a variable, then $(\forall x)A$ is a formula.

\footnote{I am thus dividing science in something of the way Hilbert divided mathematics into its real and ideal parts.}
We will use ‘$A \land B$’ as an abbreviation for $\neg (A \supset \neg B)$, ‘$A \lor B$’ for $\neg A \supset B$, and ‘$(\exists x)A$’ for $\neg (\forall x)\neg A$.

**Definition 3** An occurrence of a variable $x$ in a formula $A$ is *bound* if it occurs in a subformula of the form $(\forall x)B$; otherwise the occurrence is *free*.

Thus, for example, in $(\forall x)(x = y)$, $x$ is bound and $y$ is free.

**Substitution** of a term $t$ for a variable $x$ in a formula $A(x)$ is defined so that if a variable occurring free in $t$ would be bound after the substitution, then the bound variable is changed first to prevent this. Thus, for example, if $t$ is $x'$, then substituting $t$ for $y$ in $(\forall x)(x = y)$ results in $(\forall z)(z = x')$ and not in $(\forall x)(x = x')$.

**Logic Axiom Schemes** If $A$, $B$, and $C$ are any formulas, if $x$ is any variable, if $t$ is any term, if $A(x)$ is any formula, and if $A(t)$ is the result of substituting $t$ for $x$ in $A(x)$, then the following are axioms:

1. $A \supset (B \supset A)$.
2. $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$.
3. $(\neg A \supset \neg B) \supset (B \supset A)$.
4. $(\forall x)A(x) \supset A(t)$.
5. $C \supset (\forall x)C$ if $x$ does not occur free in $C$.
6. $(\forall x)(A \supset B) \supset ((\forall x)A \supset (\forall x)B)$.
7. $(\forall x)(x = x)$.
8. $(\forall x)(\forall y)(x = y \supset (A(x) \supset A(y)))$.

**Non-logical axioms**

1. $(\forall x)(\forall y)(x' = y' \supset x = y)$.
2. $(\forall x)\neg(x' = 0)$.
3. $(\forall x)(x + 0 = x)$.
4. $$(\forall x)(\forall y)(x + y' = (x + y)')$$.

5. $$(\forall x)(x0 = 0)$$.

6. $$(\forall x)(\forall y)(xy' = xy + x)$$.

7. If $A(x)$ is any formula and $x$ any variable, then $A(0) \land (\forall x)(A(x) \supset A(x')) \supset (\forall x)A(x)$ is a non-logical axiom.

Rules of inference

1. From $A \supset B$ and $A$ to deduce $B$.

2. From $A(x)$, where $x$ is a variable which does not occur free in any assumption on which $A(x)$ depends, to deduce $$(\forall x)A(x)$$.

Definition 4 A deduction from a set $\Gamma$ of assumptions is a sequence of formulas such that each formula is an axiom or is a formula in $\Gamma$ or else is derived from one or more previous formulas in the sequence by one of the rules of inference.

We write $\Gamma \vdash A'$ to mean that there is a deduction from the set $\Gamma$ of assumptions whose conclusion is $A$.

We write $\vdash A'$ to mean that $\Gamma \vdash A$ and $\Gamma$ is the empty set.

The following theorem can be proved about this system:

**Theorem 1** For every partial recursive function $f(x_1, \ldots, x_n)$ there is a formula $A(x_1, \ldots, x_n, y)$ such that $f(m_1, \ldots, m_n) = k$ if and only if $\vdash A(m_1^*, \ldots, m_n^*, k^*)$.

**B Gödel’s Incompleteness Theorem and its proof**

Gödel’s Incompleteness Theorem says that in any formal system which includes arithmetic that is strong enough to be “interesting,” there is a formula which, while true, cannot be proved. More precisely, it says this of any formal system in which all partial recursive functions can be represented in the sense of Theorem 1 of Appendix A, which is proved for an example of the kind of formal system to which the theorem applies.

To get a hint of the proof, imagine that we have a computer built

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22 A complete proof would be too technical for this paper.
like existing computers and working on the same principles but which has unlimited time and space (including unlimited memory) at its disposal. It is generally accepted in mathematical logic and theoretical computer science that the functions that can be computed by such a computer are precisely the partial recursive functions.\(^\text{23}\)

Now in our idealized computer, as in any existing computer, everything is coded by numbers. This was brought home to me around 1990, when I was at Concordia University in Montreal and when a colleague using a PC running MS DOS had a manuscript typed by a secretary on a Macintosh. At the time, I had (by a kind of accident) the only computer system then available in our department that could be used to copy the Macintosh file onto a PC diskette. After I did the copying and was reading the file to make certain that it had been correctly copied, I noticed that some accented characters had come out differently in the PC version of the file than they had been in the Macintosh version. The following table shows one change that I happened to notice:

<table>
<thead>
<tr>
<th>Macintosh</th>
<th>DOS</th>
</tr>
</thead>
<tbody>
<tr>
<td>é</td>
<td>A</td>
</tr>
</tbody>
</table>

This reminded me that what is actually stored in the file is only the *number* of the character, i.e. its position number in the font. The character that appears when a file is displayed on the screen or printed is determined by the operating system and the fonts it is using. This is why a file may display different characters when read by different operating systems.

Anybody with experience using a computer knows that the numerical coding allows the computer to be programmed to determine whether one has a single character, a string of characters (a word), a sequence of words, etc. Furthermore, anybody familiar with a programming language or a spreadsheet knows that the computer can be programmed to go back and forth between the character and its number.

An examination of the definitions of the example in Appendix A should make it clear to anybody with any knowledge of programming that our ide-

\(^{23}\)This set of functions can also be described as the set of functions computable by a Turing machine, or by the set of all lambda-definable functions. It is not necessary for this paper to understand how any of these classes of functions are defined; it is enough to know that they have all been proved to be the same class of functions, and this is part of the reason that each of these classes of functions is now regarded as the class of functions that can be computed by our idealized computer.
alized computer can be programmed to determine whether or not a given number is the code of a variable, the constant, a term, or a formula. It can be programmed to go back and forth between a number \( m \) and the term \( m^* \) that represents it. It can be programmed to determine whether or not a number codes a variable which occurs free or bound in the formula coded by another number. It can be programmed to determine the number of the formula resulting from the formula corresponding to a given code by the substitution of the term corresponding to a second code for a variable corresponding to a third code. This allows it to be programmed to determine whether a given number codes a logical or non-logical axiom. And it can be programmed to determine whether one number which codes a formula in a sequence of formulas coded by a second number follows from formulas that occur earlier in the sequence by one of the rules. Thus, the computer can be programmed to determine whether a number codes a valid deduction.

By these results and Theorem 1 of Appendix A, there is a formula \( G(x, y) \) with two free variables with the property that

\[
\vdash G(m^*, n^*) \iff m \text{ is the number of a formula } A(x) \text{ with one free variable and } n \text{ is the number of a proof of } A(m^*) \text{ (which results from } A(x) \text{ by substituting } m^* \text{ for } x).
\]

Now let \( A(x) \) be the formula \( (\forall y){\neg}G(x, y) \) (which has one free variable, \( x \)). Let \( g \) be the number which codes this formula. Then

\[
\vdash G(g^*, n^*) \iff g \text{ is the number of a formula } (\forall y){\neg}G(x, y) \text{ with one free variable and } n \text{ is the number of a proof of } (\forall y){\neg}G(g^*, y)
\]

But \( \vdash (\forall y){\neg}G(g^*, y) \) implies \( \vdash {\neg}G(g^*, n^*) \) for any \( n \). Hence, if the system is consistent,\(^{24}\) then

\[
{\not\vdash} (\forall y){\neg}G(g^*, y).
\]

But \( (\forall y){\neg}G(g^*, y) \) is true if and only if \( {\not\vdash} (\forall y){\neg}G(g^*, y) \). This means that \( (\forall y){\neg}G(g^*, y) \) says \( {\not\vdash} (\forall y){\neg}G(g^*, y) \). Hence, we have good reasons to believe that \( (\forall y){\neg}G(g^*, y) \) is true. This is our true but unprovable formula.

One way of looking at G"odel’s Incompleteness Theorem is that it says that for any theory strong enough to be “interesting,” it is impossible to

\(^{24}\)In most systems of formal logic, including the example of Appendix A, a contradiction implies any formula. Hence, in most inconsistent systems any formula can be proved. It is standard practice in science to replace immediately any theory found to be inconsistent, so it makes sense to assume that we are dealing with consistent theories.
give any sound and complete axiomatization. Here, an axiomatization does not necessarily have to be finite; it is sufficient that it give an effectively computable test for correctness of any supposed proof. Hence, what Gödel’s Incompleteness Theorem really says is that there is no sound and complete specification of the truth conditions of the theory. This applies, in particular, to the theory of elementary arithmetic and to any theory containing it.

On the other hand, in order to accept the proof of Gödel’s Incompleteness Theorem, we need to believe in the truths of arithmetic.

References


[2] Canadian Association of University Teachers. Report of the Committee of Inquiry on the Case Involving Dr. Nancy Olivieri, the Hospital for Sick Children, the University of Toronto, and Apotex Inc. Published by the author and available from their web site at http://www.caut.ca/english/issues/acadfreedom/olivieri.asp.


