Superrigid subgroups of solvable Lie groups

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Abstract

It is not difficult to see that every group homomorphism from $\mathbb{Z}^k$ to $\mathbb{R}^n$ extends to a homomorphism from $\mathbb{R}^k$ to $\mathbb{R}^n$. (Essentially, this is the fact that a linear transformation can be defined to have any desired action on a basis.) We will see other examples of discrete subgroups Γ of connected groups G, such that the homomorphisms defined on Γ can ("almost") be extended to homomorphisms defined on all of G.

Eg. Group homomorphism $\phi\colon \mathbb{Z}^k \to \mathbb{R}^d$
$\Rightarrow \phi$ extends to homo $\hat{\phi}\colon \mathbb{R}^k \to \mathbb{R}^d$.

Proof. Standard basis $\{e_1, \ldots, e_k\}$ of $\mathbb{R}^k$.
Let $\hat{\phi}\colon \mathbb{R}^k \to \mathbb{R}^d$ be linear,
such that $\hat{\phi}(e_i) = \phi(e_i)$.

- $\hat{\phi}$ linear
  $\Rightarrow \hat{\phi}$ is an additive homomorphism

- $\langle e_1, \ldots, e_k \rangle = \mathbb{Z}^k$
  $\Rightarrow \hat{\phi}(z) = \phi(z)$ for $z \in \mathbb{Z}^k$

Any homomorphism into $\mathbb{R}^d$
can be thought of as a homo into $\mathrm{GL}_{d+1}(\mathbb{C})$:

$$\mathbb{R}^d \cong \begin{pmatrix} 1 & 0 & 0 & \mathbb{R} \\ 0 & 1 & 0 & \mathbb{R} \\ 0 & 0 & 1 & \mathbb{R} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Representation Theory.

Prop. *Group homomorphism* $\phi\colon \mathbb{Z}^k \to \mathrm{GL}_d(\mathbb{C})$
$\Rightarrow \phi$ *virtually extends to homo* $\hat{\phi}\colon \mathbb{R}^k \to \mathrm{GL}_d(\mathbb{C})$
such that $\hat{\phi}(\mathbb{R}^k) \subset \overline{\phi(\mathbb{Z}^k)}$.
(*"Zariski closure"*)

This means $\mathbb{Z}^k$ is *superrigid* in $\mathbb{R}^k$.
"Representations defined on $\mathbb{Z}^k$
(virtually) extend to be defined on $\mathbb{R}^k$"

Generalize to other *solvable* groups: A connected subgroup G of $\mathrm{GL}_d(\mathbb{C})$ is **solvable** if it is upper triangular

$$G \subset \begin{pmatrix} \mathbb{C}^\times & \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{C}^\times & \mathbb{C} \\ 0 & 0 & \mathbb{C}^\times \end{pmatrix}$$

(or is after a change of basis).

Defn. Γ is a **lattice** in G if

- Γ is discrete and
- G/Γ is compact.

(Every element of G is a bounded distance from some element of Γ.)

$\hat{\phi}$ virtually extends ϕ:

$\exists$ finite-index subgroup Γ of $\mathbb{Z}^k$,
such that $\hat{\phi}(\gamma) = \phi(\gamma)$ for all $\gamma \in \Gamma$.

$\hat{\phi}(\mathbb{R}^k) \subset \overline{\phi(\mathbb{Z}^k)}$: "image of ϕ controls image of $\hat{\phi}$."

Eg. If all matrices in $\phi(\mathbb{Z})$ commute,
then all matrices in $\hat{\phi}(\mathbb{R})$ commute.

Eg. If all matrices in $\phi(\mathbb{Z})$ fix a vector v,
then all matrices in $\hat{\phi}(\mathbb{R})$ fix v.

Eg. If all matrices in $\phi(\mathbb{Z})$ belong to $\mathrm{GL}_d(\mathbb{R})$,
then all matrices in $\hat{\phi}(\mathbb{R})$ belong to $\mathrm{GL}_d(\mathbb{R})$.

Good properties of $\phi(\mathbb{Z})$ carry over to $\hat{\phi}(\mathbb{R})$.

Examples of lattices.

$$G = \begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} & \mathbb{R} \\ 0 & 1 & \mathbb{R} & \mathbb{R} \\ 0 & 0 & 1 & \mathbb{R} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Gamma = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 0 & 1 & \mathbb{Z} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\overline{\Gamma} = G \qquad \text{superrigid}$$

$$G = \begin{pmatrix} \mathbb{R}^+ & 0 & 0 \\ 0 & \mathbb{R}^+ & 0 \\ 0 & 0 & \mathbb{R}^+ \end{pmatrix}$$

$$\Gamma = \begin{pmatrix} 2^{\mathbb{Z}} & 0 & 0 \\ 0 & 2^{\mathbb{Z}} & 0 \\ 0 & 0 & 2^{\mathbb{Z}} \end{pmatrix}$$

$$\overline{\Gamma} = G \qquad \text{superrigid}$$

$$G = \begin{pmatrix} 1 & \mathbb{R} & \mathbb{C} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \Gamma = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z}+\mathbb{Z}i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\overline{G} = G \qquad \overline{\Gamma} = G$$

$$G' = \begin{pmatrix} 1 & t & \mathbb{C} \\ 0 & 1 & 0 \\ 0 & 0 & e^{2\pi i t} \end{pmatrix} \qquad \overline{G'} = \begin{pmatrix} 1 & \mathbb{R} & \mathbb{C} \\ 0 & 1 & 0 \\ 0 & 0 & \mathbb{T} \end{pmatrix}$$

$$\Gamma' = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z}+\mathbb{Z}i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \Gamma$$

Γ is a lattice in both G and G'.

$$\overline{\Gamma} = G \neq \overline{G'} \qquad \text{so } \overline{\Gamma} \neq \overline{G'}$$

Γ is *not* superrigid in G'.

E.g., the identity map $\phi\colon \Gamma \to \Gamma$
does not extend to homo $\hat{\phi}\colon G' \to \overline{\Gamma}$.

Proof. $\overline{\Gamma} = G$ is abelian, but G' is not abelian.

Prop. Γ *superrigid in* G

$$\Rightarrow \overline{\Gamma} = \overline{G} \qquad \left(\operatorname{mod} \overline{Z(G)}\right).$$

Converse:

Thm (Witte). *A lattice* Γ *in a solvable grp* G *is superrigid iff* $\overline{\Gamma} = \overline{G}$ $\left(\operatorname{mod} \overline{Z(G)}\right)$.

$\overline{\Gamma} \neq \overline{G'}$: some of the rotations associated to G' do not come from rotations associated to Γ

$$\operatorname{rot} \begin{pmatrix} \alpha & * \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} \dfrac{\alpha}{|\alpha|} & 0 \\ 0 & \dfrac{\beta}{|\beta|} \end{pmatrix}$$

Borel Density. Γ *latt in simply conn, solv* G
$\Rightarrow \exists$ *cpct torus* $T \subset \overline{G}$*, s.t.* $T\,\overline{\Gamma} = \overline{G}$*.*

Cor (Witte). *A lattice* Γ *in a Lie group* G *is "superrigid" iff*

- $\overline{\Gamma} = \overline{G}$ $\quad$ $\bigl(\text{mod}\, \overline{Z(G)} \cdot (\text{cpct ss normal subgrp})\bigr)$
- *and simple part of* Γ *is "superrigid."*

Thm (Margulis Superrigidity Theorem).
All lattices in $\mathrm{SL}_n(\mathbb{R})$ *are "superrigid" if* $n \geq 3$.
Similar for other simple Lie groups, $\mathbb{R}$-rank ≥ 2.

Cor (Margulis Arithmeticity Theorem).
Every lattice in $\mathrm{SL}_n(\mathbb{R})$ *is "arithmetic" if* $n \geq 3$.
(like $\mathrm{SL}_n(\mathbb{Z})$)

Only way to make a lattice: take integer points
(and minor modifications)

Similar for other simple groups with $\mathbb{R}$-rank ≥ 2.

Prop. *Group homomorphism* $\phi\colon \mathbb{Z}^k \to \mathbb{R}^d$
$\Rightarrow$ *ϕ extends to homomorphism* $\hat{\phi}\colon \mathbb{R}^k \to \mathbb{R}^d$.

Proof. $\hat{\Gamma} = \operatorname{graph}(\phi) \subset \mathbb{R}^k \times \mathbb{R}^d$ $\qquad X = \operatorname{span}\hat{\Gamma}$.

Claim. X is the graph of a function $\hat{\phi}\colon \mathbb{R}^k \to \mathbb{R}^d$.

- $\operatorname{graph}(\hat{\phi})$ is a subgroup $\Rightarrow$ $\hat{\phi}$ is a homo.
- $\operatorname{graph}(\phi) \subset \operatorname{graph}(\hat{\phi}) \Rightarrow \hat{\phi}$ extends ϕ.

Suffices to show:

1) X projects onto $\mathbb{R}^k$.
2) $X \cap (0 \times \mathbb{R}^d) = 0$.

1) $\pi_1(X) =$ conn subgrp of $\mathbb{R}^k$ that contains $\mathbb{Z}^k$
$\Rightarrow \pi_1(X) = \mathbb{R}^k$.

2) *Fact.* $(\operatorname{span}\hat{\Gamma})/\hat{\Gamma}$ is compact (for any closed $\hat{\Gamma}$).

$\pi_1|_{\operatorname{graph}(\phi)}$ is proper (because ϕ is continuous)
$+$ Fact $+$ π_1 is a homo $\Rightarrow \pi_1|_X$ is proper
$\Rightarrow X \cap \pi_1^{-1}(0)$ is compact.

Generalize. $\Gamma \subset G$, $\phi\colon \Gamma \to H$
$\stackrel{?}{\Longrightarrow}$ ϕ extends to $\hat{\phi}\colon G \to H$.

Defn. Syndetic hull of Γ:
connected subgrp $X \supset \Gamma$, such that X/Γ is cpct.

Same proof if:

- Every closed $\hat{\Gamma} \subset G \times H$ has a synd hull.
- No conn, proper subgroup of G contains Γ.
- H has no nontrivial compact subgroups.

Furthermore, syndetic hulls unique $\Rightarrow$ $\hat{\phi}$ unique.

Fact. H simply connected, solvable Lie group
$\Rightarrow$ H has no nontrivial compact subgroups.

Fact. Γ cocompact in simply conn, solvable G
$\Rightarrow$ no conn, proper subgroup of G contains Γ.

Eg. $G = \widetilde{\mathrm{SO}}(2) \ltimes \mathbb{R}^2$ $\qquad \Gamma = \mathbb{Z} \times (\mathbb{Z}, 0)$

$\mathbb{R} \times (\mathbb{R}, 0)$ is not a subgroup of G,
so Γ does not have a syndetic hull in G.

Fact. In a (simply) connected, unipotent group, syndetic hulls exist and are unique.

(*Uniqueness:* $\exp\colon \mathfrak{g} \to G$ is a diffeomorphism, so intersection of connected subgrps is connected.)

Prop (Malcev). *Suppose*

- G *and* H *are connected, unipotent groups*
- Γ *is a lattice in* G
- $\phi\colon \Gamma \to H$ *is a homomorphism.*

Then ϕ *extends uniquely to* $\hat{\phi}\colon G \to H$.

Cor (Malcev). Γ_i *lattice in conn, unipotent* G_i.

$$\Gamma_1 \cong \Gamma_2 \Rightarrow G_1 \cong G_2.$$

More generally [Saito], this is true for simply connected, **split**, solvable Lie groups, i.e.,

$$G, H \hookrightarrow \begin{pmatrix} \mathbb{R}^\times & \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{R}^\times & \mathbb{C} \\ 0 & 0 & \mathbb{R}^\times \end{pmatrix}$$

Key Fact. Γ *closed* $\subset$ *simply conn solv group* G
$\overline{\mathrm{Ad}_G \Gamma} \supset$ maximal cpct subgrp of $\overline{\mathrm{Ad} G}$
$\Rightarrow \Gamma$ *has a unique syndetic hull in* G.

Proof. By induction, $[\Gamma, \Gamma]$ has a unique syndetic hull U in $[G, G]$.

Uniqueness $\Rightarrow \Gamma \subset N_G(U)$. Wolog $G = N_G(U)$.

Mod out U, so Γ abel. Wolog $G = C_G(\Gamma)$.

I.e., $\Gamma \subset Z(G)$. Wolog $G = Z(G)$ is abelian. ∎

Need: $N_G(U), C_G(\Gamma)$, and $Z(G)$ are connected.

Lem. G *conn solvable Lie group*

- Q *Zariski-closed subgroup of* $\mathrm{GL}_n(\mathbb{R})$
- $\rho\colon G \to \mathrm{GL}_n(\mathbb{R})$ *homo, such that*
- $Q \supset$ *maximal compact subgroup of* $\overline{G^\phi}$

$\Rightarrow \rho^{-1}(Q)$ *is connected.*

Let $\rho = \mathrm{Ad}_G$, $Q = N_{\overline{\mathrm{Ad} G}}(U)$, $C_{\overline{\mathrm{Ad} G}}(\Gamma)$, e.

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