Some arithmetic groups that cannot act on the line

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Example

Example: SL(2, Z) does not act on \( \mathbb{R} \) because it has elements of finite order.

Fact

There exist other examples that act on \( \mathbb{R} \).
But all are "small". (I think all known are in SO(1, n)).

Conjecture

Large arithmetic groups (R-rank > 1) cannot act on \( \mathbb{R} \).

Bounded generation by unip subgrps

Note: Invertible matrix \( \sim \) Id by row operations.

Key fact: \( g \in \text{SL}(2, \mathbb{Z}) \sim \) Id by integer (Z) row ops.

Example

\[
\begin{array}{ccc}
13 & 31 \\
5 & 12 \\
\end{array}
\sim
\begin{array}{ccc}
1 & 2 \\
0 & 1 \\
\end{array}
\]

But # steps is not bounded: \( \mathcal{U} \) and \( \mathcal{V} \) do not boundedly generate \( \text{SL}(2, \mathbb{Z}) \).

Transformation groups

Given: group \( \Gamma \), (connected) manifold \( M \).

\( \exists \) What are the actions of \( \Gamma \) on \( M \)?

\( \text{I.e.: } \exists \text{ what are homos } \phi : \Gamma \to \text{Homeo}_+(M) \)?

Question

\( \exists \text{ faithful action?} \)

Simplest case

\[ \dim M = 1, \quad \text{so } M = S^1 \text{ or } \mathbb{R}. \]

Assume \( M = \mathbb{R} \).

Example

\( \mathbb{Z} \) acts on \( \mathbb{R} \).

\[ (T_n(x) = x + n) \implies T_{m+n} = T_m \circ T_n \]

Question

\( \exists \text{ (faithful) action of } \Gamma \) on \( \mathbb{R} \)?

Conjecture

\( \Gamma \) does not act on \( \mathbb{R} \).

Theorem (Carter-Keller-Paige, Lifschitz-Morris)

\( \Gamma \) no action on \( \mathbb{R} \) if \( \Gamma \cong \text{SL}(3, \mathbb{Z}) \) or \( \text{SL}(2, \mathbb{Z}[[\alpha]]) \)

Proof combines bdd generation and bdd orbits.

Unipotent subgroups: \( \mathcal{U} = \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}, \quad \mathcal{V} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \).

Theorem (Liehl)

\( \text{SL}(2, \mathbb{Z}[1/p]) \) bddly gen’d by elem mats.\n
\( \text{I.e., } T \sim \text{Id by } \mathbb{Z}[1/p] \) col ops, # steps is bdd.

Easy proof

Assume Artin’s Conjecture:

\( \forall r \neq \pm 1, \) perfect square,

\( \exists \) \( q \), s.t. \( r \) is primitive root modulo \( q \):

\( \{ r, r^2, r^3, \ldots \} \mod q = \{ 1, 2, 3, \ldots, q - 1 \} \)

Assume \( \exists q \) in every arith progression \( \{ a + kb \} \).

\( \exists q = a + kb, \) \( p \) is a primitive root modulo \( q \).
**Theorem (Lifschitz-Morris)**

\( \Gamma = \text{SL}(2, \mathbb{Z}[1/p]) \) acts on \( \mathbb{R} \) \( \Rightarrow \) every \( U \)-orbit bdd.

\[ U = \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \varphi = \begin{bmatrix} p & 0 \\ 0 & 1/p \end{bmatrix} \]

Assume \( U \)-orbit and \( V \)-orbit of \( x \) not bdd above. Assume \( \varphi \) fixes \( x \). (\( \varphi \) does have fixed pts, so not an issue.)

- Wolog \( \varpi(x) < \psi(x) \).
- Then \( \varphi^n(\varpi(x)) < \varphi^n(\psi(x)) \).
- LHS = \( \varphi^n(\varpi(x)) = (\varphi^n \varphi \varphi^{-n})(x) \approx x \approx \infty \).
- RHS = \( \varphi^n(\psi(x)) = (\varphi^n \varphi \varphi^{-n})(x) \approx \psi(x) < \infty \).

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**Corollary**

\( \Gamma \) cannot act on \( \mathbb{R} \).

**Proof.**

Suppose there is a nontrivial action.

It has fixed points: 

Remove them:

Take a connected component:

\( \Gamma \) acts on open interval (\( = \mathbb{R} \)) with no fixed point.

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**Bounded orbits**

\( \text{SL}(2, \mathbb{Z}[1/p]) \) bddly gen’d by elem mats. Le., \( T \sim \text{Id by } \mathbb{Z}[1/p] \) col ops, \# steps is bdd.

**Theorem (Liehl)**

\( \text{SL}(2, \mathbb{Z}[1/p]) \) bddly gen’d by elem mats.

Le., \( T \sim \text{Id by } \mathbb{Z}[1/p] \) col ops, \# steps is bdd.

**Proof.**

\[ \begin{bmatrix} a & b \\ c & d \end{bmatrix} q = a + kb \text{ prime, } p \text{ is prim root} \]

\[ \begin{bmatrix} q & b \\ * & * \end{bmatrix} p^{e} \equiv b \pmod{q}; \quad p^{e} = b + k'q \]

\[ \begin{bmatrix} q & p^{e} \\ * & * \end{bmatrix} p^{e} \text{ unit: can add anything to } q \]

\[ \begin{bmatrix} 1 & p^{e} \\ * & * \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ * & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \]

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**Corollary**

\( U \)-orbits and \( V \)-orbits are bounded.

**Proof.**

- \( Bdd \) generation: \( U \cap V \cap \cdots \cap V \).
- \( Bdd \) orbits: \( U \)-orbits and \( V \)-orbits are bounded.