Amenable groups that act on the line

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Abstract

Let G be a group. It is obvious that if G has an infinite cyclic quotient, then G has a nontrivial action on the real line by orientation-preserving homeomorphisms. The converse is not true in general, but, using an idea of E.Ghys, we prove that the converse does hold for all finitely generated, amenable groups. The proof is surprisingly easy, and combines elementary results from group theory, topology, and the theory of dynamical systems.

Theory of noncommutative dynamical systems

Given: group G, (connected) manifold M.
¿ What are the actions of G on M ?
I.e.: ¿ What are homos $\phi\colon G \to \mathrm{Homeo}_+(M)$?

Question

¿ $\exists$ (nontrivial) *action ?*

Simplest case

$\dim M = 1$, so $M = S^1$ or $\mathbb{R}$.
Assume $M = \mathbb{R}$.

Example

$\mathbb{Z}$ acts on $\mathbb{R}$.

Corollary

$G \twoheadrightarrow \mathbb{Z} \implies$ *G acts on $\mathbb{R}$.*

Converse

- *false* in general (e.g., $G \subset \widetilde{\mathrm{SL}(2,\mathbb{R})}$)
- *true* for finitely generated *amenable* groups.

Corollary

G amenable, acts faithfully on $\mathbb{R}$ (*faithful: no kernel*)
$\implies$ *every f.g. subgrp of $G \twoheadrightarrow \mathbb{Z}$.* (*$G$ is locally indicable.*)

Remark (Burns & Hale, 1972)

G loc ind, countable $\implies$ G acts faithfully on $\mathbb{R}$.

Proof of the converse (for amen grps)

Assume

G amenable, finitely generated, acts faithfully on $\mathbb{R}$.

We show $G \twoheadrightarrow \mathbb{Z}$.

Outline of Proof (4 easy steps, plus one fact).

1. G has a left-invariant order.
2. Ghys: G acts on the space $\mathcal{O}$ of left-inv't orders.
3. Amenability: $\exists$ G-inv't probability meas on $\mathcal{O}$.
4. Poincaré Recurrence: there is a recurrent order.
5. *Known:* G has a recurrent order $\implies G \twoheadrightarrow \mathbb{Z}$.

Step 1: G has a left-invariant order

Assume: G acts faithfully on $\mathbb{R}$.

Definition

$a \prec b \iff a(0) < b(0)$ or … (break ties)

Exercise

1. $\prec$ *is a total order on G.*
2. $\prec$ *is left-invariant.* ($a \prec b \implies ca \prec cb$)

Hint: orientation-pres: $x < y \implies c(x) < c(y)$

Exercise (assume G countable)

G acts faithfully on $\mathbb{R} \iff \exists$ left-inv't order on G.

Step 2: idea of Étienne Ghys

We know: G has a left-invariant order.

Proposition

$\mathcal{O} := \{$ *left-invariant orders on* $G\} \neq \emptyset$.

- $\mathcal{O} \subset 2^{G\times G}$ *is compact (& Hausdorff)*

  Basic open set: $\mathcal{O}_{a_1,\ldots,a_r} = \{ \prec \mid a_1 \prec a_2 \prec \cdots \prec a_r \}$
- G *acts on* $\mathcal{O}$ *by right translation:*

  $$a \prec_g b \iff ag^{-1} \prec bg^{-1}$$

This is an action by homeomorphisms.

Proof.

$\mathcal{O}_{a_1,\ldots,a_r}\, g = \mathcal{O}_{a_1 g,\ldots,a_r g}$. □

Step 3: Amenability

We know: G acts continuously on $\mathcal{O}$.

G *amenable*
$\implies \exists$ G-invariant probability measure on $\mathcal{O}$.

Step 4: Poincaré Recurrence Theorem

$\forall g \in G$, *a.e.* $\prec$ *is recurrent for* g:

$a_1 \prec a_2 \prec \cdots \prec a_r$
$\implies a_1 g^n \prec a_2 g^n \prec \cdots \prec a_r g^n, \quad \exists\, n \to \infty$

Exercise

a.e. $\prec$ *is recurrent for every element of* G.

Step 5: construct $G \twoheadrightarrow \mathbb{Z}$

We know: $\exists$ recurrent left-invariant order on G.

Proposition (classical)

G *has bi-invariant order*

$\implies$ G *acts on* $\mathbb{R}$, *such that* $\forall g$, *either*
$\forall x \in \mathbb{R}, g(x) \geq x$ *or* $\forall x \in \mathbb{R}, g(x) \leq x$

$g(a(0)) > a(0) \implies ga \succ a \implies g \succ e$

$\implies$ $\{$*elt's with a fixed point*$\}$ *is subgrp* (normal)

Suppose a *fixes* x, b *fixes* y, *not* x.

Say $a(y) > y$. *Then* $ab^{\pm 1}(y) = a(y) > y$.

But $ab^{\pm 1}(x) \gtrless x$.

$\implies$ G/N *acts on* $\mathbb{R}$, *no element has a fixed point.*

G/N acts on $\mathbb{R}$, no element has a fixed point.

Proposition (Hölder, 1901)

$\overline{G}$ *acts on* $\mathbb{R}$, *no el't has a fixed pt* $\implies$ $\overline{G}$ *abelian*.

Proof.

Fix $a \succ e$. We may assume $a(x) = x + 1$ (no f.p.).
So $\lfloor g(0) \rfloor = \lfloor \log_a g \rfloor$ $(a^m \preceq g \prec a^{m+1})$.
Therefore $gh(0) = g(0) + h(0) \pm 1$ ("subadditive").
Define $\varphi(g) = \lim_{n\to\infty} \dfrac{g^n(0)}{n}$.
Then φ is a homomorphism $\overline{G} \hookrightarrow \mathbb{R}$. □

So $G \twoheadrightarrow \overline{G} \twoheadrightarrow \mathbb{Z}$. ($G$ is f.g.) □

Recall: G amenable, f.g., acts on $\mathbb{R} \implies G \twoheadrightarrow \mathbb{Z}$.

Conjecture (Linnell)

Can replace amenable with no free subgroups.

Definition

Say G is a *Poincaré-Recurrence group* if:
$\forall$ (continuous) action of G on compact, metric X,
$\exists x \in X$, x is recurrent for all cyclic subgroups of G.

Questions

1. ¿ G *no free subgroups* $\implies$ G *is P-R group* ?
2. ¿ $\exists$ torsion-free *P-R group that is not amenable* ?
3. ¿ *What are the P-R groups* ?

Details or references for the proofs in this lecture

Dave Witte Morris: Amenable groups that act on the line, *Algebraic & Geom. Topology* 6 (2006) 2509–2518.
`http://arxiv.org/abs/math/0606232`

Further development of these ideas

Andrés Navas: On the dynamics of (left) orderable groups
`http://arxiv.org/abs/0710.2466`

Introduction to groups acting on 1-dimensional manifolds

Étienne Ghys: Groups acting on the circle, *Enseign. Math.* (2) 47 (2001), no. 3-4, 329–407.
`http://retro.seals.ch/digbib/view?did=c1:470307&p=373`

Study of groups with a left-invariant order

Valerii M. Kopytov and Nikolaii Ya. Medvedev: *Right-Ordered Groups*, Plenum, New York, 1996.