Minimal isotropic simple $\mathbb{Q}$-groups of higher real rank

joint work with Vladimir Chernousov and Lucy Lifschitz

University of Virginia Algebra Seminar
February 27, 2008

Abstract. Let $\mathbf{G}$ be a simple algebraic group over the field $\mathbb{Q}$ of rational numbers, and assume $\mathbf{G}$ is isotropic. It is well known that $\mathbf{G}$ must contain a subgroup that is isogenous to SL_2. Furthermore, if $\mathbb{Q}$-rank$(\mathbf{G}) > 1$, then $\mathbf{G}$ contains a subgroup that is isogenous to SL_3 or Sp_4.

The theory of lattices in semisimple Lie groups leads to interest in the case where $\mathbb{R}$-rank$(G) > 1$ (where $\mathbb{R}$ is the field of real numbers). In this setting, no finite list of possible subgroups will suffice, but joint work with L. Lifschitz and V. Chernousov shows that G must contain a subgroup that is isogenous either to SL_3 or to a restriction of scalars of SL_2 that is not absolutely simple.

The theory of arithmetic groups (or, in geometric terms, the theory of locally symmetric spaces) requires an understanding of almost-simple algebraic groups defined over $\mathbb{Q}$. For the question to be discussed in today's lecture, it suffices to work in the more elementary setting of simple Lie algebras over $\mathbb{Q}$.

Notation. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{Q}$.

Remark. Up to isomorphism, $\mathfrak{g}$ is a $\mathbb{Q}$-linear subspace of $\mathrm{Mat}_{n\times n}(\mathbb{Q})$ that is closed under the bracket operation $[X,Y] = XY - YX$.

E.g., the *adjoint representation* $\mathrm{ad}_{\mathfrak{g}} \colon \mathfrak{g} \to \mathrm{End}_{\mathbb{Q}}(\mathfrak{g})$ is defined by $(\mathrm{ad}_{\mathfrak{g}}\, x)v = [x,v]$.

Example. Some classical Lie algebras are:

- $\mathfrak{sl}_n(\mathbb{Q}) = \{\, X \in \mathrm{Mat}_{n\times n}(\mathbb{Q}) \mid \mathrm{tr}\, X = 0 \,\}$,
- $\mathfrak{sp}_{2n}(\mathbb{Q}) = \{\, X \in \mathfrak{sl}_{2n}(\mathbb{Q}) \mid XJ = -JX^T \,\}$, $J = \begin{bmatrix} 0 & \mathrm{I}_{n\times n} \\ -\mathrm{I}_{n\times n} & 0 \end{bmatrix}$
  (J is an invertible, skew-symmetric, $2n \times 2n$ matrix),
- $\mathfrak{so}_n(B;\mathbb{Q}) = \{\, X \in \mathfrak{sl}_n(\mathbb{Q}) \mid XB = -BX^T \,\}$,
  where B is an invertible, symmetric, $n \times n$ matrix.

There are also exceptional Lie algebras (of type $E_{6,7,8}$, F_4, and G_2).

Assumption. We assume $\mathfrak{g}$ is *isotropic*;
i.e., $\mathrm{ad}_{\mathfrak{g}}\, x$ is diagonalizable over $\mathbb{Q}$, for some $x \in \mathfrak{g}$.
If $\mathfrak{g} \subset \mathfrak{sl}_n(\mathbb{Q})$, this means
$\mathfrak{g}$ contains a matrix that is diagonalizable over $\mathbb{Q}$
or, equivalently, $\mathfrak{g}$ contains a nonzero nilpotent matrix.

Definition. Assume $\mathfrak{g} \subset \mathfrak{sl}_n(\mathbb{Q})$ (e.g., via the adjoint representation). Recall that $\mathrm{rank}_{\mathbb{Q}}\, \mathfrak{g}$ is the maximum dimension of a subspace of $\mathfrak{g}$ consisting of matrices that are simultaneously diagonalizable over $\mathbb{Q}$. Thus,

$$\mathfrak{g} \text{ is isotropic} \iff \mathrm{rank}_{\mathbb{Q}}\, \mathfrak{g} \geq 1.$$

Example.

- $\mathfrak{sl}_n(\mathbb{Q})$ and $\mathfrak{sp}_n(\mathbb{Q})$ are isotropic
  ($\mathrm{rank}_{\mathbb{Q}}\, \mathfrak{sl}_n(\mathbb{Q}) = n-1$ and $\mathrm{rank}_{\mathbb{Q}}\, \mathfrak{sp}_{2n}(\mathbb{Q}) = n$).
- $\mathfrak{so}_n(B;\mathbb{Q})$ is isotropic iff $vBv^T = 0$ for some nonzero $v \in \mathbb{Q}^n$.

It is well known that $\mathfrak{sl}_2(\mathbb{Q})$ is the only (isotropic, simple) Lie algebra that is minimal under inclusion:

Proposition (well known). $\mathfrak{g} \supset \mathfrak{sl}_2(\mathbb{Q})$.

Our main theorem is similar to this, in that it finds the minimal elements in a particular class of Lie algebras. The Lie algebras in the class are large, in the sense that they have large rank. Here is a classical result of this type:

Proposition (well known).
If $\mathrm{rank}_{\mathbb{Q}}\, \mathfrak{g} \geq 2$, *then* $\mathfrak{g} \supset \mathfrak{sl}_3(\mathbb{Q})$ *or* $\mathfrak{sp}(4,\mathbb{Q})$.

Sketch of proof. Note that $\mathfrak{sl}_3(\mathbb{Q})$ is the split Lie algebra of type A_2, and $\mathfrak{sp}_4(\mathbb{Q})$ is the split Lie algebra of type $C_2 = B_2$.

A classical theorem of Borel and Tits states that $\mathfrak{g}$ contains a simple subalgebra $\mathfrak{h}$, such that $\mathrm{rank}_{\mathbb{Q}}\, \mathfrak{h} = \mathrm{rank}_{\mathbb{Q}}\, \mathfrak{g}$, and $\mathfrak{h}$ is $\mathbb{Q}$-split. Then $\mathfrak{g}$ contains a $\mathbb{Q}$-split subalgebra of rank 2; i.e., of type A_2, B_2, or G_2. By inspection, the root system G_2 contains A_2. □

In short, among the Lie algebras of rank ≥ 2, exactly two are minimal under inclusion.

In arithmetic and differential geometry, it is important to understand the behavior of $\mathfrak{g}$ when it is tensored with $\mathbb{R}$. Let us ignore the very few real Lie algebras of rank 1:

$\{\, \mathfrak{so}(1,n), \mathfrak{su}(1,n), \mathfrak{sp}(1,n), F_{4,1} \,\} \times \{\text{compact}\}$.

Assumption. Assume $\operatorname{rank}_{\mathbb{R}} \mathfrak{g} \geq 2$.

Remark. The assumption that $\operatorname{rank}_{\mathbb{R}} \mathfrak{g} \geq 2$ is common in the theory of arithmetic groups. For example, letting Γ be the integer points of the corresponding (simply connected) algebraic group:

- The Margulis Superrigidity Theorem describes the representation theory of Γ.
- Serre conjectured that the *Congruence Subgroup Property* is true for Γ.
- It is hoped that Γ has *bounded generation*.
- It is conjectured that Γ has no nontrivial, orientation-preserving actions on the real line, and today's theorem is a key ingredient in some progress toward a proof. That is the topic of tomorrow's colloquium talk.

Example. Let $\mathfrak{g} = \mathfrak{sl}_2(K)$, with $K = \mathbb{Q}[\sqrt{3}]$. Then

$$\mathfrak{g} \supset \left\{ \begin{bmatrix} x + \sqrt{3}y & 0 \\ 0 & -(x + \sqrt{3}y) \end{bmatrix} \right\}$$

This is 2-dim'l subalgebra is diagonalizable over $\mathbb{R}$, so $\operatorname{rank}_{\mathbb{R}} \mathfrak{g} = 2$. Indeed,

$\mathfrak{g} \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{sl}_2(\mathbb{R})$.

Example. $\mathfrak{sl}_2\big(\mathbb{Q}(i)\big) \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathfrak{sl}_2(\mathbb{C})$, so $\operatorname{rank}_{\mathbb{R}} \mathfrak{sl}_2\big(\mathbb{Q}(i)\big) = 1$.

Proposition. *Let* $\mathfrak{g} = \mathfrak{sl}_2(K)$. *Then:*

(1) $\mathfrak{g} \otimes_{\mathbb{Q}} \mathbb{R} \cong$ *product of* $\mathfrak{sl}_2(\mathbb{R})$*'s and* $\mathfrak{sl}_2(\mathbb{C})$*'s,*

(2) K *a* proper *extension of* $\mathbb{Q}$*, not imaginary quadratic*
$\implies \operatorname{rank}_{\mathbb{R}} \mathfrak{g} \geq 2$.

Remark. Conversely, $\mathfrak{g}$ simple, $\mathfrak{g} \otimes_{\mathbb{Q}} \mathbb{R} \cong$ product of $\mathfrak{sl}_2(\mathbb{R})$'s and $\mathfrak{sl}_2(\mathbb{C})$'s $\implies \mathfrak{g} \cong \mathfrak{sl}_2(K)$.

The above examples include most of the minimal Lie algebras:

Theorem (Chernousov-Lifschitz-Morris).
$\exists$ *simple Lie subalgebra* $\mathfrak{h}$ *of* $\mathfrak{g}$*, such that* $\mathfrak{h} \otimes_{\mathbb{Q}} \mathbb{R} \cong$

(1) *product of* $\mathfrak{sl}_2(\mathbb{R})$*'s and* $\mathfrak{sl}_2(\mathbb{C})$*'s* (*with* ≥ 2 *factors*)*, or*

(2) $\mathfrak{sl}_3(\mathbb{R})$*, or*

(3) $\mathfrak{sl}_3(\mathbb{C})$.

Remark. In case (2), we have $\mathfrak{h} \cong \mathrm{SL}_3(\mathbb{Q})$ or $\mathfrak{su}_3(x_1 x_3^\tau + x_2 x_2^\tau; K)$,
where K is a real quadratic extension of $\mathbb{Q}$,
and τ is the Galois automorphism of K over $\mathbb{Q}$.
There is a similar description in case (3).

Remark.

(1) The theorem was previously proved by Margulis and Raghunathan (independently) under the assumption that some minimal parabolic $\mathbb{R}$-subalgebra of $\mathfrak{g}$ is defined over $\mathbb{Q}$.

(2) There is also an analogue for number fields other than $\mathbb{Q}$.

Here is a concise (but less detailed) statement of the theorem:

Corollary. $\mathfrak{g}$ *contains an isotropic, simple Lie subalgebra* $\mathfrak{h}$*, such that* $\operatorname{rank}_{\mathbb{R}} \mathfrak{h} \geq 2$*, and* $\mathfrak{h}$ *is* ***quasisplit****.*

Method of proof of the theorem. The *Tits Classification* provides a "list" of the simple Lie algebras (over any field), and they are considered case-by-case. The subalgebra $\mathfrak{h}$ is constructed explicitly. □

Most of the classical cases are fairly straightforward.

Example. $\mathfrak{so}_4(x_1 x_2 - x_3^2 + a x_4^2; \mathbb{Q}) \cong \mathfrak{sl}\big(2, \mathbb{Q}[\sqrt{a}]\big)$.

Proof. Let

$$V = \{\, \text{Hermitian mats in } \mathrm{Mat}_{2\times 2}\big(\mathbb{Q}(\sqrt{a})\big) \,\} = \left\{ \begin{bmatrix} x & y + \sqrt{a}z \\ y - \sqrt{a}z & w \end{bmatrix} \,\middle|\, x, y, z, w \in \mathbb{Q} \right\}.$$

This is a 4-dimensional vector space over $\mathbb{Q}$.

$\mathrm{SL}\big(2, \mathbb{Q}[\sqrt{a}]\big)$ acts on V by $H \mapsto AH\overline{A}^T$,

and this action preserves the determinant

$xw - (y + \sqrt{a}z)(y - \sqrt{a}z) = xw - (y^2 - az^2) = x2 - y^2 + az^2$,

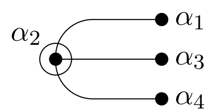
so this provides an embedding

$\mathrm{SL}\big(2, \mathbb{Q}[\sqrt{a}]\big) \hookrightarrow \mathrm{SO}(x_1 x_2 - x_3^2 + a x_4^2; \mathbb{Q})$.

Comparing dimensions yields the desired conclusion. □

There are some interesting results among the non-classical types. They are obtained constructing a subalgebra of $\mathfrak{g} \otimes_{\mathbb{Q}} \mathbb{C}$, and showing that it is defined over $\mathbb{Q}$.

Proposition. *Type* ${}^{3,6}D_{4,1}$ *contains* $\mathfrak{sl}_2(K)$*, with* $|K : \mathbb{Q}| = 4$*.*



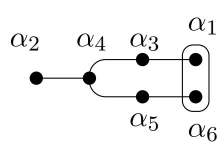
Sketch of proof. Let $\mu = 2\alpha_2 + \alpha_1 + \alpha_3 + \alpha_4$ be the maximal root. Then $\alpha_1, \alpha_3, \alpha_4, \mu$ are mutually orthogonal, and the subalgebra they span is defined over $\mathbb{Q}$. Unfortunately, it is not simple.

Conjugating by the Weyl reflection corresponding to α_2 yields the mutually orthogonal roots

$$\alpha_2 + \alpha_1, \alpha_2 + \alpha_3, \alpha_2 + \alpha_4, \alpha_2 + \alpha_1 + \alpha_3 + \alpha_4.$$

For an appropriate choice of maximal torus (defined over $\mathbb{Q}$), the subalgebra they (and their negatives) generate is defined over $\mathbb{Q}$ and is $\mathbb{Q}$-simple. (The action of the Galois group on the roots is known explicitly.) □

Proposition. *Type* ${}^2E_{61}^{29}$ *contains an isotropic subalgebra of type* ${}^2A_5$*.*

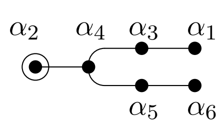


Sketch of proof. $\mathfrak{h}$ is generated by the roots

$$\alpha_3, \alpha_1, \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5, \alpha_6, \alpha_5.$$

(It is the centralizer of the subalgebra generated by $\pm\alpha_2$.) □

Proposition. *Type* ${}^2E_{61}^{35}$ *contains an isotropic triality subalgebra (i.e., an isotropic simple subalgebra of type* ${}^3D_{4,1}$ *or* ${}^6D_{4,1}$*).*



Sketch of proof. $\mathfrak{h}$ is generated by the roots

$$\alpha_2,\ \alpha_1 + \alpha_3 + \alpha_4,\ \alpha_3 + \alpha_4 + \alpha_5,\ \alpha_4 + \alpha_5 + \alpha_6.$$

(The action of the Galois group is used to prove that it is defined over $\mathbb{Q}$.) □

Remark. Let $\mathbf{G}$ be the (simply connected) almost-simple algebraic $\mathbb{Q}$-group corresponding to $\mathfrak{g}$. If $\mathbf{G}$ is of type E_6, the subgroup H is simply connected, but, unfortunately, *not* if $\mathbf{G}$ is a triality group.

Remark. The theorem assumes $\mathfrak{g}$ is isotropic, and constructs a subalgebra that is isotropic.

- For geometric applications, it would be interesting to find the simple Lie algebras that are minimal among the anisotropic ones (that have real rank ≥ 2).
- It might also be interesting to find anisotropic subalgebras of the isotropic ones.

Reference

Vladimir Chernousov, Lucy Lifschitz, and Dave Witte Morris: Almost-minimal nonuniform lattices of higher rank, *Michigan Mathematical Journal* (to appear).
http://arxiv.org/abs/0705.4330