Locally symmetric subspaces of locally symmetric spaces

joint work with Vladimir Chernousov and Lucy Lifschitz

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Abstract. It has long been known that only two manifolds are minimal in the category of symmetric spaces X = G/K of rank greater than 1. (We assume G is a connected, semisimple Lie group with no compact factors.) Namely, every symmetric space in this category contains either the product of two hyperbolic planes or the symmetric space associated to SL(3,R). The corresponding problem for noncompact spaces of finite volume that are locally symmetric, rather than symmmetric, was solved in joint work with Lucy Lifschitz and Vladimir Chernousov, but infinitely many manifolds are minimal in this category. The compact case remains open.

Symmetric spaces

Remark. Hyperbolic space $\mathbb{H}^n$ contains $\mathbb{H}^k$ for $k < n$.
So the unique minimal hyperbolic space is $\mathbb{H}^2$.

Note that $\widetilde{X} = \mathbb{H}^n$ is a *symmetric space*:

1. isometry group $G \doteq \mathrm{SO}(1,n)$ is transitive, so $\mathbb{H}^n = G/K$;
2. for each $x \in \mathbb{H}^n$, $\exists$ isometry φ fixing x, and $D\varphi_x = -\mathrm{I}$.

(Actually, (1) is redundant.)

Example.

- $\mathbb{R}^n$ is a symmetric space.
  Ignore: we assume $\widetilde{X} \neq \mathbb{R}^n \times \widetilde{X}'$. (*no flat factors*)
- S^n is a symmetric space.
  Ignore: we assume $\widetilde{X} \neq \mathrm{cpct} \times \widetilde{X}'$ (*noncompact type*)

Proposition (well known). *$\mathbb{H}^2$ is the only minimal symm space: $\widetilde{X}$ symmetric $\implies$ $\widetilde{X} \supset \mathbb{H}^2$ (as a totally geodesic subspace)*

We can do better if $\widetilde{X}$ has rank 2;
i.e., $\mathbb{R}^2 \subset X$ (totally geodesic) — a *flat.*

Example. $\widetilde{X}_1 \times \widetilde{X}_2 \supset \mathbb{H}^2 \times \mathbb{H}^2$.

Proposition (well known). rank $\widetilde{X} \geq 2 \implies \widetilde{X} \supset$

- $\mathbb{H}^2 \times \mathbb{H}^2$ *or*
- $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3) = \left\{ \begin{array}{c} 3 \times 3 \text{ symmetric, positive-definite} \\ \text{matrices of determinant } 1 \end{array} \right\}$.

Remark. Assuming rank $\widetilde{X} \geq 2$ rules out only

- real hyperbolic space $\mathbb{H}^n$,
- complex hyperbolic space $\mathbb{C}\mathbb{H}^n$,
- quaternionic hyperbolic space $\mathbb{H}\mathbb{H}^n$, and
- the Cayley plane $\mathbb{O}\mathbb{H}^2$ defined from the octonions.

Locally symmetric spaces

Definition. X *locally symmetric*: universal cover $\widetilde{X}$ is symmetric.

Exercise. X is locally symmetric
$\iff \forall x \in X$, $\exists$ isometry φ of open neigh of x, s.t. $D\varphi_x = -\mathrm{I}$ and X is complete.

Assumption. We assume

- X is irreducible i.e., $X \neq X_1 \times X_2$ (up to finite cover), and
- X has finite volume, but is *not* compact.

Example. X = hyperbolic n-mfld of finite volume. $(X = \mathbb{H}^n/\Gamma)$
$\implies X \supset$ image of $\mathbb{H}^2$, but this image may not be closed in X.
If X is "arithmetic," one can show $X \dot{\supset} \mathbb{H}^2/\mathrm{SL}(2,\mathbb{Z})$.
(I do not know anything about embeddings of **non**arithmetic hyperbolic manifolds.)

Assumption. rank $\widetilde{X} \geq 2$.

Example. $\Gamma = \mathrm{SL}\big(2, \mathbb{Z}[\sqrt{2}]\big)$. $\quad ((a + \sqrt{2}b)^\sigma = a - \sqrt{2}b)$
$\Gamma \cong \{\, (\gamma, \gamma^\sigma) \,\} \subset \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \doteq \mathrm{Isom}(\mathbb{H}^2 \times \mathbb{H}^2)$
Let $X = (\mathbb{H}^2 \times \mathbb{H}^2)/\Gamma$.

Remark. Similar construction (with $\sqrt{2} \mapsto$ other algebraic int ω) provides loc symm spaces modeled on product of $\mathbb{H}^2$s and $\mathbb{H}^3$s
$$X = (\mathbb{H}^2 \times \mathbb{H}^2 \times \cdots \times \mathbb{H}^2) \times (\mathbb{H}^3 \times \mathbb{H}^3 \times \cdots \times \mathbb{H}^3)/\,\mathrm{SL}\big(2, \mathbb{Z}[\omega]\big)$$
because $\mathrm{SL}(2,\mathbb{C}) \doteq \mathrm{SO}(1,3) \doteq \mathrm{Isom}(\mathbb{H}^3)$.

This infinite family includes all but two of the symmetric spaces with locally symmetric quotients that are minimal:

Theorem (Chernousov-Lifschitz-Morris). *Assume*

- X = *irred, noncpct, complete, locally symm space of finite vol*
- $\operatorname{rank} \widetilde{X} \geq 2$.

Then X contains an irreducible, closed, totally geodesic, noncompact, locally symmetric submanifold that is modeled on either:

- *a product of $\mathbb{H}^2$s and $\mathbb{H}^3$s (with ≥ 2 factors), or*
- *the symmetric space*
$$\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3) \cong \left\{ \begin{matrix} 3\times 3 \text{ symmetric, positive-definite} \\ \text{matrices of determinant } 1 \end{matrix} \right\},$$
or
- *the symmetric space*
$$\mathrm{SL}(3,\mathbb{C})/\mathrm{SU}(3) \cong \left\{ \begin{matrix} 3\times 3 \text{ positive-definite, Hermitian} \\ \text{matrices of determinant } 1 \end{matrix} \right\}.$$

Although this theorem is geometric, the first step in the proof is to turn it into a problem in algebra, by using fundamental theorems about locally symmetric spaces of higher rank:

Theorem (Mostow-Margulis Rigidity). $\Gamma_1 \dot{\supset} \Gamma_2 \Rightarrow X_1 \dot{\supset} X_2$.

Theorem (Margulis Arithmeticity Theorem).
Γ is an arithmetic subgroup of G: $\Gamma \doteq G \cap \mathrm{SL}(\ell,\mathbb{Z}) = G_{\mathbb{Z}}$
for some embedding $G \dot{\hookrightarrow} \mathrm{SL}(\ell,\mathbb{R})$ with $G_{\mathbb{Q}}$ dense in G.

We call $G_{\mathbb{Q}}$ a $\mathbb{Q}$-*form* of G.

Corollary. *It suffices to show $G_{\mathbb{Q}}$ contains either:*

- $\mathrm{SL}(2,F)$, *for some extension F of $\mathbb{Q}$ (not imag quad), or*
- *some ("isotropic") $\mathbb{Q}$-form of* $\mathrm{SL}(3,\mathbb{R})$, *or*
- *some ("isotropic") $\mathbb{Q}$-form of* $\mathrm{SL}(3,\mathbb{C})$.

Theorem (classical). *There is a "list" of all the $\mathbb{Q}$-forms $G_{\mathbb{Q}}$:*

- $\mathrm{SL}(n,F)$, *or*
- $\mathrm{Sp}(n,F)$
- $\mathrm{SO}(B;F)$, *or*
- $\mathrm{SU}(B,\tau;F)$, *or*
- $\mathrm{SL}(n,D)$, *where D is a division algebra over F, or*
- $\mathrm{SU}(B,\tau;D)$, *where D is a quaternion algebra over F, or*
- *a few other "exceptional" or "triality" cases.*

The proof considers these possible $\mathbb{Q}$-forms case-by-case.

Example. $\mathrm{SL}(n,D) \supset \mathrm{SL}(n, \text{max'l subfield } K) \supset \mathrm{SL}(2,K) \neq \mathrm{SL}(2,\mathbb{Q})$.

Example. $\mathrm{SO}(x_1x_2 - x_3^2 + ax_4^2) \dot{\supset} \mathrm{SL}\big(2,\mathbb{Q}[\sqrt{a}]\big)$.

Proof. Let
$$V = \{\, \text{Hermitian mats in } \mathrm{Mat}_{2\times 2}\big(\mathbb{Q}(\sqrt{a})\big) \,\} = \left\{ \begin{bmatrix} x & y+\sqrt{a}z \\ y-\sqrt{a}z & w \end{bmatrix} \;\middle|\; x,y,z,w \in \mathbb{Q} \right\}.$$

This is a 4-dimensional vector space over $\mathbb{Q}$.

$\mathrm{SL}\big(2,\mathbb{Q}[\sqrt{a}]\big)$ acts on V by $H \mapsto AH\overline{A}^T$,
and this action preserves the determinant
$$xw - (y+\sqrt{a}z)(y-\sqrt{a}z) = xw - (y^2 - az^2) = x2 - y^2 + az^2,$$
so this provides an embedding
$$\mathrm{SL}\big(2,\mathbb{Q}[\sqrt{a}]\big) \hookrightarrow \mathrm{SO}(x_1x_2 - x_3^2 + ax_4^2;\mathbb{Q}).$$
□

Remark. We assume X is *not* compact (and construct a subspace that is not compact).

- It would be interesting to find the locally symmetric spaces that are minimal among the compact ones (that are irreducible and have higher rank).
- It might also be interesting to find compact subspaces of the noncompact ones.

Remark. The theorem has a concise formulation in algebraic terms: If $\mathbf{G}$ is a connected, isotropic, almost-simple algebraic $\mathbb{Q}$-group, with $\mathbb{R}$-rank $\mathbf{G} \geq 2$, then $\mathbf{G}$ contains a $\mathbb{Q}$-subgroup $\mathbf{H}$ with the same properties, such that $\mathbf{H}$ is quasi-split.

This was previously known in some special cases:

- If $\mathbb{Q}$-rank $\mathbf{G} \geq 2$, it follows from the existence of a $\mathbb{Q}$-split almost-simple subgroup of the same $\mathbb{Q}$-rank [Borel-Tits].
- It was proved by Margulis and Raghunathan (independently) under the assumption that some minimal parabolic $\mathbb{R}$-subgroup of $\mathbf{G}$ is defined over $\mathbb{Q}$.

There is also an analogue for number fields other than $\mathbb{Q}$.

References

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Appendices

Proposition (well known). $\operatorname{rank} \widetilde{X} \geq 2$, $\widetilde{X}$ *irreducible* $\Longrightarrow$ $\widetilde{X} \supset \mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ *or* $\mathrm{Sp}(4,\mathbb{R})/\mathrm{SU}(2)$.

- $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3) \cong \left\{ \begin{array}{c} 3\times 3 \text{ positive-definite, symmetric} \\ \text{matrices of determinant } 1 \end{array} \right\}$.
- $\mathrm{SL}(3,\mathbb{C})/\mathrm{SU}(3) \cong \left\{ \begin{array}{c} 3\times 3 \text{ positive-definite, Hermitian} \\ \text{matrices of determinant } 1 \end{array} \right\}$.
- $\mathrm{Sp}(4,\mathbb{R})/\mathrm{SU}(2) \cong \left\{ \begin{array}{c} 2\times 2 \text{ symmetric matrices over } \mathbb{C}, \\ \text{with determinant } 1, \\ \text{and positive-definite imaginary part} \end{array} \right\}$

$$= \text{“Siegel upper half space of genus 2.”}$$

Remark. $\mathrm{Sp}(4,\mathbb{R}) = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \;\middle|\; \begin{bmatrix} A^T B = B^T A \\ C^T D = D^T C \\ A^T D - B^T C = I \end{bmatrix} \right\}$ acts by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \circ M = (AM + C)(BM + D)^{-1}.$$

Constructing examples of symmetric spaces.

Example.
- $G = \mathrm{SL}(2,\mathbb{R})$,
- $K = \mathrm{SO}(2) = \{\, g \in G \mid gg^T = \mathrm{I} \,\} = \{\, g \in G \mid \alpha(g) = g \,\}$ where $\alpha(g) = (g^{-1})^T \in \operatorname{Aut} G$.
- $\mathbb{H}^2 \cong G/K \cong \left\{ \begin{array}{c} 2\times 2 \text{ symmetric, positive-definite} \\ \text{matrices of determinant } 1 \end{array} \right\}$
$= \left\{ \begin{bmatrix} x+y & z \\ z & x-y \end{bmatrix} \;\middle|\; \begin{array}{c} x^2 - y^2 - z^2 = 1 \\ x > 0 \end{array} \right\}$,
G acts by $M \mapsto gMg^T$.

Proposition.
- $G =$ *connected Lie group* (simple $\times \cdots \times$ simple)
- $\alpha \in \operatorname{Aut} G$, $\alpha^2 = 1$,
- $K = \{\, g \in G \mid \alpha(g) = g \,\}$ *compact*

$\Longrightarrow$ G/K *is symmetric.*

Example.
- $G = \mathrm{SL}(3,\mathbb{R})$,
- $K = \mathrm{SO}(3) = \{\, g \in G \mid gg^T = \mathrm{I} \,\} = \{\, g \in G \mid \alpha(g) = g \,\}$ where $\alpha(g) = (g^{-1})^T \in \operatorname{Aut} G$.
- $G/K \cong \left\{ \begin{array}{c} 3\times 3 \text{ symmetric, positive-definite} \\ \text{matrices of determinant } 1 \end{array} \right\}$,
G acts by $M \mapsto gMg^T$.

Theorem (well known). *The $\mathbb{Q}$-forms of* $\mathrm{SL}(3,\mathbb{R})$ *that lead to* ***noncompact*** *locally symmetric spaces are*
- $\mathrm{SL}(3,\mathbb{Q})$, *and*
- $\mathrm{SU}_3(x_1 x_2^\tau - x_3 x_3^\tau; F)$, *where F is a real quadratic ext'n of $\mathbb{Q}$, and τ is the Galois automorphism of F over $\mathbb{Q}$.*

A similar result holds for $\mathrm{SL}(3,\mathbb{C})$: in the conclusion, replace $\mathbb{Q}$ with all possible imaginary quadratic extensions K of $\mathbb{Q}$ (and F is not real).