Survey of invariant orders on arithmetic groups

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Abstract

At present, there are more questions than answers about the existence of an invariant order on an arithmetic group. (By "order," we mean a transitive binary relation $\prec$, such that $x \prec y \Rightarrow y \nprec x$.) We will discuss four different versions of the problem: the order may be required to be total, or allowed to be only partial, and the order may be required to be invariant under multiplication on both sides, or only on one side. One version is trivial, but the other three are related to interesting conjectures in the theory of arithmetic groups.

Γ = arithmetic group

Example

$\Gamma = \mathrm{SL}(2,\mathbb{Z})$ or $\mathrm{SL}(n,\mathbb{Z})$
or $\mathrm{SL}(n,\mathbb{Z}[\sqrt{2}])$

In general: G Lie group (connected), e.g., $\mathrm{SL}(n,\mathbb{R})$
$\Gamma = G(\mathbb{Z})$
(Need to assume technical conditions.)

Γ = arithmetic group (or finitely generated grp)

Question

$\exists$ invariant order $\prec$ on Γ ?

- *total* $(x \prec y$ or $x \succ y$ or $x = y)$
  or partial
- *left-invariant* $(x \prec y \implies ax \prec ay,\ \forall x,y,a)$
  or bi-invariant *(also invariant on the right)*

Not many answers yet (for Γ arithmetic).

Example

$\mathbb{Z}$ has a bi-invariant total order (namely, $<$).

Left-invariant partial orders

Proposition

Γ has lots of left-invariant partial orders
(unless Γ is torsion).

Proof.

Fix $g \in \Gamma$ (∞ order). Let $P = \{\, g^n \mid n > 0 \,\}$. (or other semigrp $\not\ni e$)

Define $x \prec y \iff x^{-1}y \in P$.

- transitive: $x \prec y$ & $y \prec z$
  $\implies x^{-1}z = (x^{-1}y)(y^{-1}z) \in P$
- left-invariant: $x \prec y$
  $\implies (ax)^{-1}(ay) = x^{-1}y \in P$ □

Bi-invariant total orders

Proposition

Γ has bi-invariant total order $\implies \Gamma \twoheadrightarrow \mathbb{Z}$. (if Γ f.g.)
I.e., Γ is indicable.

Corollary. Every (f.g.) subgrp of Γ is indicable.
I.e., Γ is *locally* indicable.

Theorem (Kazhdan et al.)

Γ indicable arith grp $\implies G \doteq \mathrm{SO}(1,n)$ or $\mathrm{SU}(1,n)$.
(Group with Kazhdan's property (T) is not indicable.)

Corollary. Usually no bi-inv't total order on Γ (arith).

Conj? [Rolfsen?] exist (virtually) for $\mathrm{SO}(1,3)$? $\mathrm{SO}(1,n)$?
I have no idea for $\mathrm{SU}(1,n)$. (some are indicable)

Bi-invariant partial orders

Recall: Usually no bi-invariant total order on Γ (arith).

I believe no (nontrivial) bi-invariant *partial* order
unless $\operatorname{rank}_{\mathbb{R}} G = 1$
(i.e. $G = \mathrm{SO}(1,n)$ or $\mathrm{SU}(1,n)$ or $\mathrm{Sp}(1,n)$ or $F_{4,1}$)

Equivalent:

- Every normal semigroup in Γ is a subgroup.
- $\forall g \in \Gamma$, e is a product of conjugates of g.

Known for G (and sometimes for $\mathbb{Q}$-points of G).

Problem: Prove for $\Gamma = \mathrm{SL}(3,\mathbb{Z})$. (**\$100**)

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Theorem

$\operatorname{rank}_{\mathbb{R}} G = 1$
$\implies \Gamma$ *(relatively) hyperbolic*
$\implies \exists$ *quasimorphism* $\Gamma \to \mathbb{Z}$ [Epstein-Fujiwara]
$\implies \exists$ *normal semigroup that is not a subgroup*
$\implies \exists$ *bi-invariant partial order.*

Definition (quasimorphism)

$\varphi \colon \Gamma \to \mathbb{Z}$ (unbdd), $\varphi(\gamma_1) + \varphi(\gamma_2) - \varphi(\gamma_1\gamma_2)$ is bdd.

Exercise. Stabilize: $\overline{\varphi}(\gamma) = \lim \varphi(\gamma^n)/n$.
$\overline{\varphi}(\lambda^{-1}\gamma\lambda) = \overline{\varphi}(\gamma)$.
$\{\, \gamma \in \Gamma \mid \overline{\varphi}(\gamma) > C \,\}$ is normal semigroup.

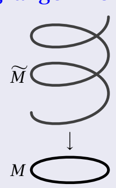
Left-invariant total order

Motivation

Γ has a left-invariant total order $\implies$

- Γ has a faithful action on $\mathbb{R}$ (iff)
- Group ring $\mathbb{Z}[\Gamma]$ has no zero divisors
  (conjectured for all torsion-free groups)
- $\mathbb{Z}[\Gamma] \cong \mathbb{Z}[\Lambda] \implies \Gamma \cong \Lambda$ [Lagrange-Rhemtulla]
- If $\Gamma = \pi_1(M)$:

$$\begin{array}{ccc} \widetilde{M} & \hookrightarrow & M \times \mathbb{R} \\ \downarrow & \swarrow & \\ M & & \end{array}$$



(iff)

One way to find a left-invariant total order

Example

$\Gamma =$ free group $\doteq \mathrm{SL}(2,\mathbb{Z}) \doteq \mathrm{SO}(1,3)_{\mathbb{Z}}$
$\implies$ every (f.g.) subgrp of Γ maps onto $\mathbb{Z}$.
I.e., Γ is locally indicable.

Conjecture (Thurston)

$\Gamma =$ *arith subgrp of* $\mathrm{SO}(1,n) \implies \dot{\Gamma}$ *indicable.*
(virtually: finite-index subgroup)

Theorem (Millson et al.)

True for $n \neq 3, 7$.

Proposition (Burns-Hale 1972)

Γ *locally indicable* $\implies \Gamma$ *has left-inv't total order.*

Conjecture (Thurston)

$\Gamma =$ *arith subgrp of* $\mathrm{SO}(1,n) \implies \dot{\Gamma}$ *indicable.*

Proposition (Burns-Hale 1972)

Γ *locally indicable* $\implies \Gamma$ *has left-inv't total order*

Rough idea of proof.

For $x, y \in \Gamma$, choose $\varphi_{x,y} \colon \langle x, y \rangle \twoheadrightarrow \mathbb{Z}$.
Define $x \prec y$ if $\varphi(x) < \varphi(y)$.

Transitive: Given $x \prec y \prec z$, assume (Zorn's Lemma)
$\varphi_{x,y} =$ restriction of $\varphi_{x,y,z}$ to $\langle x, y \rangle$. □

Conjecture (1990's)

$\exists$ *left-inv't total order on arith* $\Gamma \implies \operatorname{rank}_{\mathbb{R}} G = 1$.
(I.e., $G = \mathrm{SO}(1,n)$ *or* $\mathrm{SU}(1,n)$ *or* $\mathrm{Sp}(1,n)$ *or* $F_{4,1}$*.)*

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Theorem (Chernousov-Lifschitz-Morris, 2008)

If true for noncocompact in $\mathrm{SL}(3,\mathbb{R})$ *and* $\mathrm{SL}(3,\mathbb{C})$,
then true for noncocompact in all G.

Open question: ¿ $\exists$ *cocompact* arith group such that
no finite-index subgrp has left-inv't total order ?

Note: lattices in $\mathrm{Sp}(1,n)$ and $F_{4,1}$ are Kazhdan.

Open question: ¿ $\exists$ left-orderable Kazhdan group ?

$SL(2, \mathbb{Z}[1/p])$ has no left-invariant total order

Theorem (Lifschitz-Morris)

$\Gamma \doteq SL(2, \mathbb{Z}[1/p])$ *acts on* $\mathbb{R}$ $\Rightarrow$ *every* $\overline{U}$*-orbit bdd.*

$$\overline{u} = \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix}, \quad \underline{v} = \begin{bmatrix} 1 & 0 \\ v & 1 \end{bmatrix}, \quad \not{p} = \begin{bmatrix} p & 0 \\ 0 & 1/p \end{bmatrix}$$

Assume $\overline{U}$-orbit and $\underline{V}$-orbit of x not bdd above.
Assume $\not{p}$ fixes x. ($\not{p}$ does have fixed pts, so not an issue.)

- Wolog $\overline{u}(x) < \underline{v}(x)$.
- Then $\not{p}^n(\overline{u}(x)) < \not{p}^n(\underline{v}(x))$.
- LHS $= \not{p}^n(\overline{u}(x)) = (\not{p}^n \overline{u} \not{p}^{-n})(x) \to \overline{\infty}(x) \to \infty$.
- RHS $= \not{p}^n(\underline{v}(x)) = (\not{p}^n \underline{v} \not{p}^{-n})(x) \to \underline{0}(x) < \infty$.

$\rightarrow\leftarrow$ □

Summary

Γ = (irreducible) arithmetic group (in semisimple group G)
Assume $\operatorname{rank}_{\mathbb{R}} G \geq 2$.

Exercise

Γ *has lots of left-invariant partial orders.* (*semigroups*)

Proposition

Γ *does not have a bi-invariant total order.* ($\Gamma \not\twoheadrightarrow \mathbb{Z}$)

Conjecture

Γ *has neither:*

- *left-inv't total order* (*completely open for cocpct*), *nor*
- *bi-invariant partial order* (*completely open*).

Left-invariant total orders on lattices:

- D. W. Morris and Lucy Lifschitz: Bounded generation and lattices that cannot act on the line, Pure and Applied Mathematics Quarterly 4 (2008) 99–126. arXiv:math/0604612
- D. W. Morris: Can lattices in $SL(n, \mathbb{R})$ act on the circle? (to appear). arXiv:0811.0051
- S. Boyer, D. Rolfsen, and B. Wiest: Orderable 3-manifold groups, Ann. Inst. Fourier, Grenoble 55 (2005) 243–288. arXiv:math/0211110

Bi-invariant partial orders on Γ*:*

- D. B. Epstein and K. Fujiwara: The second bounded cohomology of word-hyperbolic groups. Topology 36 (1997), no. 6, 1275–1289.

Bi-invariant partial orders on G*:*

- D. Witte: Products of similar matrices. Proc. Amer. Math. Soc. 126 (1998) 1005–1015.

$\dot{\Gamma} \twoheadrightarrow \mathbb{Z}$:

- A. Lubotzky: Eigenvalues of the Laplacian, the first Betti number and the congruence subgroup problem. Ann. of Math. (2) 144 (1996), no. 2, 441–452. MR1418904 (98h:22013)

Orders on general groups:

- A. M. W. Glass: Partially Ordered Groups, World Scientific, River Edge, NJ, 1999. ISBN 981-02-3493-7. MR1791008 (2001g:06002)
- V. M. Kopytov and N. Ya. Medvedev: Right-ordered groups. Consultants Bureau, New York, 1996. ISBN: 0-306-11060-1 MR1393199 (97h:06024a)
- R. B. Mura and A. Rhemtulla: Orderable groups. Marcel Dekker, Inc., New York-Basel, 1977. MR0491396 (58 #10652)