Introduction to Arithmetic Groups

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Abstract. We will discuss a few basic properties of \textit{arithmetic groups}, which are certain groups of \( n \times n \) matrices with integer entries. By definition, the subject combines algebra (group theory and matrices) with number theory (the integers), but it also has connections with other areas, including the theory of periodic tilings. To learn more about these important groups, download a free copy of my book from http://arxiv.org/src/math/0106063/anc/

My book about these important groups, download a free copy of my theory (the integers), but it also has connections with other matrices with integer entries. By definition, the subject \textit{“arithmetic groups,”} which are certain groups of \( n \times n \) matrices with integer entries. 

- **Example**
  - **Symmetries of a tessellation** (periodic tiling)
  - **symmetry group** \( \Gamma = \mathbb{Z}^2 \)
  - \( \Gamma \cong \mathbb{Z}^2 \)

  **Thm** (Bieberbach, 1910). \( \forall \) tess of \( \mathbb{R}^n, \Gamma \cong \mathbb{Z}^n \).

Other spaces yield groups that are more interesting.

- **Tess’s of symmetric spaces** (e.g., \( \mathbb{R}^n, \) sphere, \( \mathbb{H}^2 \)).

  **My book:** larger isometry groups (e.g., \( \text{SL}(n, \mathbb{R}) \)).

  **Example** (classical, \( \leq \) E. Cartan, 1926)
  - **\( \text{SL}(n, \mathbb{R}) \) isometry group of a symmetric space** \( \mathcal{H}_n \).

  **My book:** systems of symmetries of \( \mathcal{H}_n \) and other non-Euclidean, non-compact symmetric spaces.

  **Theorem** (A. Borel, 1963)
  - **Every symmetric space has a tessellation** (\( \sim \)ly many \( \Gamma \)).

  **Theorem** (Margulis Arithmeticity Thm, \( \sim \)1975)
  - **List of all possibilities for** \( \Gamma \) if \( n \geq 3 \).

  - **\( \Gamma \) is grp of mats with \( \mathbb{Z} \) entries (“arithmetic grp”)**

  - **Thm** (Borel). Every symm space has a tessellation.

  **Thm.** \( \text{SL}(n, \mathbb{R}) \) has a lattice subgroup.

  **Proof** \( (n=2) \).
  - **Quaternions** \( \mathbb{H} = \{ x + iy + zj + wk \} \).
    - \( i^2 = j^2 = k^2 = -1 \), \( ij = ji \).
  - \( \mathbb{H}_3 = \{ \text{same, except} \ t^2 = 3 = k^2 \} = \text{Mat}_{2x2}(\mathbb{R}) \).
  - \( 1 = [\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}] \), \( i = [\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}] \), \( j = [\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}] \), \( k = [\begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array}] \).
  - \( \Gamma = \{ g \in \mathbb{H}_3(\mathbb{Z}) \mid \text{det} \ g = 1 \} \subset \text{SL}(2, \mathbb{R}) \).
    - \{Z-pts\} of vector space has no acc pts \( \Rightarrow \) \( \Gamma \) discrete.
    - Every pt v.s. is within bdd dist of a Z-pt, \( \Rightarrow \) every pt of \( G \) is within bdd dist of \( \Gamma \).

Other spaces yield groups that are more interesting.

- **\( \mathbb{R}^n \) is a symmetric space:**
  - **homogeneous:**
    - \( \forall x, y, \exists \) isometry \( x \rightarrow y \).
      (preserves distances)
    - reflection through a point \( (x' = -x) \) is an isometry.

  **Example**
  - **Sphere** \( \Gamma = \text{finite} \) \( \approx \) \{e\}
  - **Hyperbolic plane** \( \mathbb{H}^2 \) (upper half plane) (Poincaré disk)

  **Theorem** (A. Borel, 1963)
  - **Every symmetric space** \( X \) has a tessellation \( \sim \)ly many \( \Gamma \).

  **Lemma**
  - **\( X \) has a tess \( \iff \) \( G = \text{Isom}(X) \) has lattice subgrp \( \Gamma \):
    - \( \Gamma \) is discrete (no acc pts).
    - every el’t of \( G \) is within a bdd distance of \( \Gamma \).

  **Proof.**
  - (\( \Rightarrow \)) Fix \( x_0 \in X \).
  - For \( x_n \in \Gamma x_0 \) (discrete set), \( T_n = \{ x \in X \mid x_n \text{ is closest point of } \Gamma x_0 \} \).
  - This is a tessellation. (Voronoi diagram)

**Example**
- **\( \mathbb{H}_3(\mathbb{Z}) \) is a lattice in** \( \text{SL}(2, \mathbb{R}) \).

**Observe**
- **\( \mathbb{H}_3(\mathbb{Q}) \) is a division algebra** (every nonzero el’t has inverse).

**Proof.**
- For \( g = x + iy + zj + wk \in \mathbb{H}_3(\mathbb{Q}) \),
  - \( \text{det} g = (x + \sqrt{3}y)(x - \sqrt{3}y) - (z + \sqrt{3}w)(z - \sqrt{3}w) = x^2 - 3y^2 + z^2 - 3w^2 = g \bar{g} \)
  - \( \neq 0 \) for \( x, y, z, w \in \mathbb{Q} \) (not all 0).

Replace \( \mathbb{H}_3(\mathbb{Q}) \) with larger division alg \( D \) over \( \mathbb{Q} \).

Then \( D(\mathbb{Z}) \) is a lattice in \( \text{SL}(n, \mathbb{R}) \) (if \( D(\mathbb{R}) \cong \text{Mat}_{n\times n}(\mathbb{R}) \)).