Hamiltonian checkerboards

Dave Witte Morris
University of Lethbridge, Alberta, Canada
http://people.uleth.ca/~dave.morris
Dave.Morris@uleth.ca

Abstract. Place a checker on some square of an \( m \times n \) rectangular checkerboard. Asking whether the checker can tour the board, visiting all of the squares without repeats, is the same as asking whether a certain graph has a hamiltonian path (or hamiltonian cycle). The question becomes more interesting if we allow the checker to step off the edge of the board. This modification leads to numerous open problems, and also to connections with ideas from elementary topology and group theory. Some of the problems may be easy, but many have resisted attack for 30 years. No advanced mathematical training will be needed to understand most of this talk.

A checker is in the Southwest corner of an \( m \times n \) checkerboard.

Can the checker tour the board?

- The checker (rook?) moves North, South, East, West (not diagonally!)
- A tour must visit each square exactly once
  \textit{and return to the starting point} (hamiltonian cycle).

Yes if \( mn \) is \textbf{even}. ✓

No if \( mn \) is \textbf{odd}.

Proof.

NORTH = SOUTH and EAST = WEST.
TOTAL = NORTH + SOUTH + EAST + WEST
  \( = (2 \times \text{NORTH}) + (2 \times \text{EAST}) \)
is an even number.
TOTAL = \# squares on the checkerboard = \( mn \) is odd.

Allow the checker to step off the edge of the board.
(The board is now toroidal, rather than flat.)

Proposition

A checker can tour any board if allowed to step off the edge.

Square checkerboards \((n \times n)\) are \textbf{hamiltonian}.

Proposition

\textit{The} \( m \times n \) \textit{checkerboard is not hamiltonian if} \( \gcd(m, n) \neq 1 \).

Proof by contradiction.

The tour must have \( mn \) steps: \( E + N = mn \).
\( E \) is divisible by \( m \).
\( N \) is divisible by \( n \), so \( E \) is also divisible by \( n \).
Therefore \( E \) is divisible by \( \text{lcm}(m, n) = mn \).
So \( E \) is either \( mn \) or 0.
So the tour either never leaves one row,
or never leaves one column. \( \square \)

Definition

A board is \textbf{hamiltonian} if it has such a tour.
**Theorem (R. A. Rankin, Trotter-Erdős, Curran)**

The $m \times n$ checkerboard is hamiltonian if and only if $mx + ny = mn$ for some $x, y \in \mathbb{Z}^{>0}$ with $\gcd(x, y) = 1$.

**Proof ($\Rightarrow$, Stephen Curran).**

The board is a torus, so the path traced out by the checker is a closed path on the torus — a torus knot.

Let $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ be the knot class of this knot.

(The knot wraps $x$ times longitudinally, and wraps $y$ times meridionally.)

In other words, the checker steps off:

- the East edge of the board $x$ times, and
- the North edge of the board $y$ times.

The tour has $mx$ steps East, and $ny$ steps North.

Therefore, $mx + ny = mn$.

Since $(x, y)$ is a knot class, $\gcd(x, y) = 1$.

**Change the rules**: A tour must visit each square exactly once but need not return to the starting place. ("ham path")

**Observation**

Every checkerboard has a hamiltonian path.

**Proof.**

**Where can hamiltonian paths end?** (starting in SW corner)

**Theorem**

On an $n \times n$ (square) checkerboard:

- Hamiltonian paths always end on the main diagonal.
- $n$ even $\Rightarrow$ $\exists$ ham path to anywhere on main diagonal.
- $n$ odd $\Rightarrow$ only to every other vertex.

**Proposition**

Only certain squares are the starting point of a ham path, and only certain squares are the ending point of a ham path.

**Example:**

**Problem**

What squares can be the starting point of a ham path on an $m \times n$ projective checkerboard? (Solved for $m = n$.)

**Remark**: Starting points known $\Rightarrow$ ending points known.
Another topology: Klein bottle

- glue top to bottom without a twist (like torus), and
- glue left to right with a twist (like projective plane).

**Problem**

Find starting points and ending points of Hamiltonian paths on a Klein checkerboard.

Even the square \((n \times n)\) boards have not been studied.

**Exercise**

Every 3-dimensional checkerboard has a Hamiltonian path (starting in Southwest corner of bottom level).

What are the endpoints of Hamiltonian paths in a 3D checkerboard?

**Conjecture**

If \(\gcd(\ell, m, n) = 1\) (and \(\ell, m, n \geq 2\)), then Hamiltonian paths in the \(\ell \times m \times n\) checkerboard can end anywhere.

**Remark**

In a \(n \times n \times n\) cube, Hamiltonian paths end anywhere on the “diagonal” \(\{(x, y, z) \mid x + y + z \equiv -1 \pmod{n}\}\)

**Group theorist’s point of view**

**Definition**

- \(A\) = (finite) abelian group; e.g., \(A = \mathbb{Z}_m \oplus \mathbb{Z}_n\).
- \(S\) = generating set; e.g., \(S = \{(1, 0), (0, 1)\}\).

Cayley digraph \(\text{Cay}(A; S)\) is a digraph:

- vertices are elements of \(A\),
- directed edge \(v \rightarrow v + s\), for each \(s \in S\).

**Observation**

- \(\text{Cay}(\mathbb{Z}_m \oplus \mathbb{Z}_n; \{(1, 0), (0, 1)\})\) has a (directed) Hamiltonian cycle \(\iff\) \(m \times n\) checkerboard is Hamiltonian.
- \(\text{Cay}(\mathbb{Z}_p \oplus \mathbb{Z}_q; \{(1, 0), (0, 1), (0, 0, 1)\})\) has a (directed) Hamiltonian cycle.