TWO LECTURES ON BOUNDED COHOMOLOGY
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DAVE WITTE MORRIS,
UNIVERSITY OF LETHBRIDGE
http://people.uleth.ca/~dave.morris/

Recall. $H^*(\Gamma;\mathbb{R})$

- homogeneous n-cochain:
  $c\colon \Gamma^{n+1} \to \mathbb{R}$, Γ-invariant $\quad c(gg_0,\dots,gg_n) = c(g_0,\dots,g_n)$
- coboundary $\delta\colon C^n(\Gamma) \to C^{n+1}(\Gamma)$:

$$\delta c(g_0,\dots,g_{n+1}) = \sum_{i=0}^{n+1} (-1)^i c(g_0,\dots,\widehat{g_i},\dots,g_{n+1})$$

- $H^n(\Gamma) = Z^n(\Gamma)/B^n(\Gamma)$.

Definition (bounded cohomology)**.** $H_b^*(\Gamma;\mathbb{R})$
Same as $H^*(\Gamma;\mathbb{R})$, but require all cochains to be <u>bounded</u> functions on Γ^*.

Remark. For calculations, may prefer to use <u>inhomogeneous</u> cochains:

- $c\colon \Gamma^n \to \mathbb{R}$
- coboundary $\delta\colon C^n(\Gamma) \to C^{n+1}(\Gamma)$:

$$\delta c(g_0,\dots,g_n) = c(g_1,\dots,g_n) + \sum_{i=1}^{n} (-1)^i c(g_0,\dots,g_{i-1}g_i,\dots,g_n) \pm c(g_0,\dots,g_{n-1})$$

Example ($H_b^0(\Gamma)$)**.**

- $H^0(\Gamma) = \{\, \Gamma\text{-invariants in } \mathbb{R} \,\} = \mathbb{R}$.
- $H_b^0(\Gamma) = \{\,$<u>bounded</u> elements of $\mathbb{R}\,\} = \mathbb{R}$.

Example ($H_b^1(\Gamma)$)**.**

- $H^1(\Gamma) = \{\, \text{homomorphisms } c\colon \Gamma \to \mathbb{R} \,\}$
- $H_b^1(\Gamma) = \{\,$<u>bounded</u> homomorphisms $c\colon \Gamma \to \mathbb{R}\,\} = 0$.

We are only interested in $H_b^n(\Gamma)$ for $n \ge 2$. These groups are not easy to calculate:

Open problem (in 2006)**.** Choose a countable group Γ, such that $\exists n$, $H_b^n(\Gamma) \neq 0$. Calculate $H_b^*(\Gamma)$.

Example. It is known that:

$$H_b^n(F_2 \text{ (free group)}) = \begin{cases} \infty\text{-dimensional} & n = 2,3 \\ \langle \textit{open problem} \rangle & n > 3. \end{cases}$$

Proposition. *If A is abelian (or amenable), then $H_b^*(A) = 0$.*

Proof. *A amenable* $\implies$ $\exists$ (left) A-invariant mean on $\ell^\infty(A)$:

- $\mu\colon \ell^\infty(A) \to \mathbb{R}$ linear function (finitely additive),
- $\int_A$ positive $d\mu \geq 0$,
- $\int_A 1\, d\mu = 1$.

Given $c \in Z_b^3(\Gamma)$, let $\overline{c}(g_1, g_2) = \int_A c(x, g_1, g_2)\, d\mu(x)$.
Since $\delta c = 0$, we know $c(g_1, g_2, g_3) = c(x, g_1, g_2) - c(x, g_1, g_3) + c(x, g_2, g_3)$.
So

$$\begin{aligned} c(g_1, g_2, g_3) &= \int_A c(g_1, g_2, g_3)\, d\mu(x) \\ &= \int_A \big(c(x, g_1, g_2) - c(x, g_1, g_3) + c(x, g_2, g_3)\big) \\ &= \overline{c}(g_1, g_2) - \overline{c}(g_1, g_3) + \overline{c}(g_2, g_3) \\ &= \delta\overline{c}(g_1, g_2, g_3) \\ &= 0 \quad \text{in } H_b^2(\Gamma). \end{aligned}$$

□

Remark.

- For topological group G, define $H_b^*(G)$ using bounded cochains that are continuous.
- For (connected) Lie groups, calculation of $H_b^*(G)$ reduces to case where G is semisimple:

  $e \to R \to G \to \overline{G} \to e$ with R solvable (so amenable) and $\overline{G}$ semisimple.

Open problem (in 2006)**.** G = connected, semisimple Lie group with finite center.
Is comparison map $H_b^*(G; \mathbb{R}) \to H^*(G; \mathbb{R})$ always an isomorphism?
Remark: can fail for nontrivial coefficients.

[*Aside*] Can also define bounded cohomology of a topological space X.

Definition. Cochain in $C^*(X)$ is bounded if it is a bounded function on the space of singular simplices.

Theorem (Gromov, Brooks)**.** $H_b^*(X) = H_b^*(\pi_1(X))$.

Theorem (Thurston)**.** *M closed manifold of negative curvature*
$\implies$ *for $n \geq 2$, comparison map $H_b^n(M; \mathbb{R}) \to H^n(M; \mathbb{R})$ is surjective.*

Remark. Comparison map can fail to be injective (even in negative curvature).

AN APPLICATION OF BOUNDED COHOMOLOGY TO ACTIONS ON THE CIRCLE

Definition (classical)**.** Suppose $\rho\colon \Gamma \to \mathrm{Homeo}^+(S^1)$.
Each $g \in \Gamma$ lifts to $\widetilde{g} \in \mathrm{Homeo}^+(\mathbb{R})$ (not unique — differ by el't of $\pi_1(S^1) = \mathbb{Z}$).
$e \to \mathbb{Z} \to \widetilde{\Gamma} \to \Gamma \to e$ (central extension)
Yields $c \in H^2(\Gamma;\mathbb{Z})$ "Euler class of the action"
Euler class is trivial iff ρ lifts to $\widetilde{\rho}\colon \Gamma \to \mathrm{Homeo}^+(\mathbb{R})$.

Remark (explicit description of the Euler class)**.** Fix a lift $\widetilde{g}$ for each $g \in \Gamma$.
Choose a basepoint in $\mathbb{R}$ (say, 0), and let

$$c(g,h) = \widetilde{g}(\widetilde{h}(0)) - \widetilde{gh}(0) \in \mathbb{Z}.$$

This (inhomogeneous) 2-cocycle represents the Euler class.

Proposition (Ghys)**.** *The Euler class is represented by a <u>bounded</u> 2-cocycle.*

Proof. Choose $\widetilde{g}$ with $\widetilde{g}(0) \in [0,1)$. So $\widetilde{g}([0,1)) \subset [0,2)$.
Then

$$|c(g,h)| = \left|\widetilde{g}(\widetilde{h}(0)) - \widetilde{gh}(0)\right| \le 2 + 1 = 3.$$

□

Proposition (Ghys)**.** *The construction yields a well-defined <u>bounded Euler class</u> in $H^2_b(\Gamma;\mathbb{Z})$.*

Proof. Exercise. □

Proposition (Ghys)**.** *The <u>bounded</u> Euler class is trivial iff Γ has a fixed point in S^1.*

Proof. ($\Leftarrow$) We may assume the fixed point is the basepoint $\overline{0} \in S^1$.
Then we may choose $\widetilde{g}$ with $\widetilde{g}(0) = 0$. So $c(g,h) = 0$ for all g, h.

($\Rightarrow$) We have $c(g,h) = \varphi(gh) - \varphi(g) - \varphi(h)$ for some bounded $\varphi\colon \Gamma \to \mathbb{Z}$.
Let $\widehat{g}(x) = \widetilde{g}(x) + \varphi(g)$, so

- $\widehat{g}\,\widehat{h} = \widehat{gh}$, so $\widehat{\Gamma}$ is a lift of Γ to $\mathrm{Homeo}^+(\mathbb{R})$, and
- $|\widehat{g}(0)| \le |\widetilde{g}(0)| + |\varphi(g)| \le 1 + \|\varphi\|_\infty$.

The $\widehat{\Gamma}$-orbit of 0 is a bounded subset of $\mathbb{R}$, so it has a supremum.
This supremum is a fixed point of $\widehat{\Gamma}$ in $\mathbb{R}$; its image in S^1 is a fixed point of Γ. □

Corollary. *Suppose $H^2_b(\Gamma;\mathbb{Z}) = 0$.*
Then every orientation-preserving action of Γ on S^1 has a fixed point. □

Corollary. *Suppose*

- $H^2_b(\Gamma;\mathbb{R}) = 0$,
- $H^1(\Gamma;\mathbb{R}) = 0$, *and*
- Γ *is finitely generated.*

Then every orientation-preserving action of Γ on S^1 has a finite orbit.

Proof. The short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{T} \to 0$$

yields a long exact sequence of bounded cohomology:

$$H^1_b(\Gamma;\mathbb{T}) \to H^2_b(\Gamma;\mathbb{Z}) \to H^2_b(\Gamma;\mathbb{R}).$$

By assumption, the group at the right end is 0.

Thus, the bounded Euler class is the coboundary of some (bounded) 1-cocycle $\alpha\colon \Gamma \to \mathbb{T}$. I.e., α is a homomorphism to $\mathbb{T}$. Since $H^1(\Gamma;\mathbb{R}) = 0$ (and Γ is finitely generated), we know α is trivial on some finite-index subgroup Γ' of Γ.

Then the bounded Euler class $\delta\alpha$ is trivial on Γ', so Γ' has a fixed point p. Since Γ' has finite index, then the Γ-orbit of p is finite. □

Corollary. *Suppose*

- $H^2_b(\Gamma;\mathbb{R}) = 0$,
- $H^1(\Gamma';\mathbb{R}) = 0$, *for every finite-index subgroup* Γ' *of* Γ, *and*
- Γ *is finitely generated.*

Then Γ *has no faithful, orientation-preserving* C^1 *action on* S^1. □

Theorem (Ghys, Burger-Monod)**.** *Let* Γ *be any lattice in* $\mathrm{SL}(n,\mathbb{R})$, *with* $n \geq 3$. *Then:*

(1) *Every orientation-preserving action of* Γ *on* S^1 *has a finite orbit.*
(2) Γ *has no faithful, orientation-preserving* C^1 *action on* S^1.

Outline of proof. Ghys used a very different approach, but Burger-Monod applied bounded cohomology. They showed (in a more general setting) that the comparison map $H^2_b(\Gamma;\mathbb{R}) \to H^2(\Gamma;\mathbb{R})$ is injective. Since it is known that $H^2(\Gamma;\mathbb{R}) = 0$ (if n is sufficiently large), we conclude that $H^2_b(\Gamma;\mathbb{R}) = 0$.

The other hypotheses of the corollary are well known to be true. □

Conjecture. *Let* Γ *be any lattice in* $\mathrm{SL}(n,\mathbb{R})$, *with* $n \geq 3$. *Then:*

(1) *In any orientation-preserving action of* Γ *on* S^1, *every orbit is finite.*
(2) Γ *has no faithful, orientation-preserving* C^0 *action on* S^1.

(The conjecture is open even for $n = 3$, but partial results are known.)

COMPARING $H^2_b(\Gamma;\mathbb{R})$ WITH $H^2(\Gamma;\mathbb{R})$

To calculate $H^2_b(\Gamma;\mathbb{R})$, we would like to understand the kernel of the comparison map $H^2_b(\Gamma) \to H^2(\Gamma)$.

Let c be an (inhomogeneous) bounded 2-cocycle, and assume c is trivial in $H^2(\Gamma)$. Then $c = \delta\alpha$, for some $\alpha \in C^1(\Gamma)$. Thus, for all $g,h \in \Gamma$, we have

$$|\alpha(gh) - \alpha(g) - \alpha(h)| = |\delta\alpha(g,h)| = |c(g,h)| \leq \|c\|_\infty \text{ is bounded.}$$

Thus, α is almost a homomorphism — the error from being a homomorphism is bounded (as a function of g and h).

Definition.

- A function $\alpha\colon \Gamma \to \mathbb{R}$ is a quasimorphism if $\alpha(gh) - \alpha(g) - \alpha(h)$ is bounded.
- Let $\mathrm{QMorph}(\Gamma,\mathbb{R})$ be the space of quasimorphisms from Γ to $\mathbb{R}$.

- Let $\mathrm{NHom}(\Gamma, \mathbb{R})$ (the "near homomorphisms") be the space of functions $\Gamma \to \mathbb{R}$ that are within a bounded distance of a homomorphism.
  Note that $\mathrm{NHom}(\Gamma, \mathbb{R}) \subset \mathrm{QMorph}(\Gamma, \mathbb{R})$.

Proposition. *The kernel of the comparison map* $H_b^2(\Gamma) \to H^2(\Gamma)$ *is* $\dfrac{\mathrm{QMorph}(\Gamma, \mathbb{R})}{\mathrm{NHom}(\Gamma, \mathbb{R})}$.

Proof. We already know that $\mathrm{QMorph}(\Gamma, \mathbb{R})$ maps onto the kernel of the comparison map, via $\alpha \mapsto \delta\alpha$.

Now suppose $\alpha \in \mathrm{QMorph}(\Gamma, \mathbb{R})$, and $\delta\alpha$ is trivial in $H_b^2(\Gamma)$.
Then there is a bounded function $c \colon \Gamma \to \mathbb{R}$, such that $\delta\alpha = \delta c$.
Then $\delta(\alpha - c) = 0$, so $\alpha - c$ is a homomorphism.
So α is withing a bounded distance of a homomorphism; i.e., $\alpha \in \mathrm{NHom}(\Gamma, \mathbb{R})$. □

It is now easy to prove a fact that was mentioned earlier in these lectures.

Proposition. $H_b^2(F_2)$ *is infinite-dimensional.*

Proof. We wish to construct lots of quasimorphisms (that are not homomorphisms).

As a warm-up, let us recall there is an obvious homomorphism φ_a:

$\varphi_a(x)$ = the (signed) number of occurrences of a in the reduced representation of x.

For example, $\varphi_a(a^2ba^3b^{-3}a^{-7}b^2) = 2 + 3 - 7 = -2$.

There is an analogous homomorphism φ_b, and every homomorphism $F_2 \to \mathbb{R}$ is a linear combination of these.

Similarly, we can define a quasimorphism φ_{ab}:

$\varphi_{ab}(x)$ = the (signed) number of disjoint occurrences of ab in the reduced representation of x.

For example, $\varphi_{ab}(a^2ba^3b^{-3}a^{-7}b^2) = 1 - 1 = 0$.

Exercise: Verify that φ_w is a quasimorphism, for any reduced word w.

Exercise: Verify that φ_{a^k} $(k \geq 2)$ is not within a bounded distance of the linear span of $\{\varphi_b, \varphi_a, \varphi_{a^{k+1}}, \varphi_{a^{k+2}}, \varphi_{a^{k+3}}, \ldots\}$.
Hint: Find a word w such that $\varphi_{a^k}(w)$ is large, but the others vanish on w. □

References

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