Almost-Minimal Lattices of Higher Rank

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Remark. Hyperbolic space $\mathbb{H}^n$ contains $\mathbb{H}^k$ for $k < n$.
So the unique minimal hyperbolic space is $\mathbb{H}^2$.

Example.
$X =$ compact (arithmetic) hyperbolic n-manifold
$(X = \mathbb{H}^n/\Gamma)$
$\implies X \supset$ image of $\mathbb{H}^2$,
but this image may not be closed in X.
One can show that X contains a compact, totally geodesic submanifold of dimension 2 or 3.

Thus, the minimal manifolds in this category are the 2-dimensional ones and (some of) the 3-dimensional ones.

*Example (*more similar to the result we will be discussing*).*

$X =$ **non**compact arithmetic hyperbolic n-mfld of finite volume
$\implies X \supset$ noncpct arith hyperbolic *surface* of finite vol
(as a totally geodesic submanifold)
$\implies X \mathrel{\dot\supset} \mathbb{H}^2/\operatorname{SL}(2,\mathbb{Z})$

(By $\dot\supset$, we mean containment up to commensurability;
i.e., up to finite covers.)

Summary. In the category of noncompact arithmetic hyperbolic manifolds of finite volume, the only minimal one (up to finite covers) is $\mathbb{H}^2/\operatorname{SL}(2,\mathbb{Z})$.
That settles the arithmetic locally symmetric spaces modelled on $\mathbb{H}^n$.
The remainder of this talk will consider locally symmetric spaces modelled on other symmetric space (of noncompact type, with no flat factors)

Motivation.

- minimal examples are the most basic ("simplest") ones
- they are the base cases for induction on dimension
- sometimes, a theorem true for a subspace is automatically true for the entire space.

Proposition. *The same is true:*
noncompact, arithmetic locally symmetric space of finite volume
$\dot\supset \mathbb{H}^2/\operatorname{SL}(2,\mathbb{Z})$.

Proof. This is a geometric translation of a well-known algebraic fact:
$\mathbf{G}$ semisimple algebraic $\mathbb{Q}$-group with $\mathbb{Q}$-rank $\mathbf{G} \geq 1$
$\implies G \mathrel{\dot\supset} \operatorname{SL}(2,\mathbb{Q})$. $\square$

Henceforth, we disregard $\mathbb{H}^n$, and the other spaces of rank 1
(hyperbolic spaces over $\mathbb{C}$, the quaternions, or the octonions).
Equivalently, assume the universal cover $\widetilde{X}$ contains a 2-dim'l flat.

Theorem (Mostow Rigidity Theorem).
X is determined by the isomorphism class of $\pi_1(X) = \Gamma$.

Theorem (Margulis Arithmeticity Theorem).
X must be arithmetic determined by the isomorphism class of $\pi_1(X) = \Gamma$.

(We always assume X is irreducible
i.e., $X \neq X_1 \times X_2$; i.e., $\Gamma \mathrel{\dot{\neq}} \Gamma_1 \times \Gamma_2$.)

Example. X modelled on $\mathbb{H}^m \times \mathbb{H}^n$
$\implies X \mathrel{\dot\supset} (\mathbb{H}^2 \times \mathbb{H}^2)/\Gamma'$ or $(\mathbb{H}^2 \times \mathbb{H}^3)/\Gamma'$ or $(\mathbb{H}^3 \times \mathbb{H}^3)/\Gamma'$.

Proposition. *X modelled on a product of ≥ 2 $\mathbb{H}^k$'s*
(e.g., $\mathbb{H}^2 \times \mathbb{H}^2 \times \mathbb{H}^3 \times \mathbb{H}^3 \times \mathbb{H}^7 \times \cdots \mathbb{H}^n$)
$\implies X \mathrel{\dot\supset} (\mathbb{H}^2 \times \mathbb{H}^2 \times \cdots \times \mathbb{H}^2) \times (\mathbb{H}^3 \times \mathbb{H}^3 \times \cdots \times \mathbb{H}^3)/\Gamma'$
(each factor is either $\mathbb{H}^2$ or $\mathbb{H}^3$,
and there is more than one factor)
i.e., $\Gamma \mathrel{\dot\supset} \operatorname{SL}\big(2,\mathbb{Z}[\alpha]\big)$ $(= \Gamma')$
where α is irrat alg'ic integer (not imaginary quadratic)

Remark. The deck transformations can be described explicitly:
For each Galois conjugate $\widetilde{\alpha}$ of α, we have $\Gamma' \cong \operatorname{SL}\big(2,\mathbb{Z}[\widetilde{\alpha}]\big)$, and $\operatorname{SL}\big(2,\mathbb{Z}[\widetilde{\alpha}]\big)$ embeds naturally in

- $\operatorname{SL}(2,\mathbb{R}) \mathrel{\dot\cong} \operatorname{SO}(1,2) \mathrel{\dot\cong} \operatorname{Isom}(\mathbb{H}^2)$ if $\widetilde{\alpha}$ is real;

- $\mathrm{SL}(2,\mathbb{C}) \mathrel{\dot{\cong}} \mathrm{SO}(1,3) \mathrel{\dot{\cong}} \mathrm{Isom}(\mathbb{H}^3)$ if $\widetilde{\alpha}$ is complex.

Γ' acts on each factor of the product $\mathbb{H}^2 \times \cdots \times \mathbb{H}^3$ via one of these embeddings.

Remark. An algebraic condition determines whether or not X is minimal:
X fails to be minimal iff $\mathbb{Q}(\widetilde{\alpha})$ contains a subfield $F \neq \mathbb{Q}$,
s.t. either $F \subset \mathbb{R}$ or $|F : \mathbb{Q}| > 2$.

There is only one additional minimal example (up to finite covers), among all the locally symmetric manifolds:

Theorem (Lifschitz-Morris-Chernousov). *Assume*

- X *= irred, noncpct, complete, locally symm space of finite vol*
- $\operatorname{rank} X \geq 2$.

Then X contains a closed, totally geodesic, noncompact, locally symmetric submanifold that is modelled on either:

- *a product of $\mathbb{H}^2$s and $\mathbb{H}^3$s (with ≥ 2 factors), or*
- *the symmetric space*
$$\mathrm{SL}(3,\mathbb{R})/\,\mathrm{SO}(3) \cong \left\{ \begin{matrix} 3\times 3 \textit{ symmetric, positive-definite} \\ \textit{matrices of determinant } 1 \end{matrix} \right\},$$
or
- *the symmetric space*
$$\mathrm{SL}(3,\mathbb{C})/\,\mathrm{SU}(3) \cong \left\{ \begin{matrix} 3\times 3 \textit{ positive-definite, Hermitian} \\ \textit{matrices of determinant } 1 \end{matrix} \right\}.$$

Remark. In algebraic terms, the conclusion of the theorem means that Γ contains a finite-index subgroup of either

- some $\mathrm{SL}\big(2,\mathbb{Z}[\alpha]\big)$, or
- some (noncocompact) lattice in $\mathrm{SL}(3,\mathbb{R})$.

Remark. Some cases of the theorem were already well known (in algebraic form):

- $\widetilde{X}$ reducible (i.e., $\widetilde{X} = \widetilde{X}_1 \times \cdots \times \widetilde{X}_r$)
  $\implies X \mathrel{\dot{\supset}}$ (product of $\mathbb{H}^2$s and $\mathbb{H}^3$s)$/\Gamma'$
  (with r factors)
- $\mathbb{Q}$-$\operatorname{rank} X \geq 2$ (i.e., $X \mathrel{\dot{\supset}}$ 2-dim'l closed, simply connected flat)
  $\implies \Gamma \mathrel{\dot{\supset}} \mathrm{SL}(3,\mathbb{Z})$ or $\mathrm{Sp}(4,\mathbb{Z})$
  (and $\mathrm{Sp}(4,\mathbb{Z}) \mathrel{\dot{\supset}} \mathrm{SL}\big(2,\mathbb{Z}[\sqrt{2}]\big)$)

Idea of proof. There is a list of all of the (arithmetic) lattices. Because of the cases that were already known, we need only those whose $\mathbb{Q}$-rank is 1 and have an irreducible universal cover (i.e., G is simple):

- $\mathrm{SL}(2,\mathbb{Z})$,
- $\mathrm{SO}(B;\mathbb{Z})$,
- $\mathrm{SU}\big(B,\tau;\mathbb{Z}[\sqrt{d}]\big)$
- or replace $\mathbb{Z}$ with $\mathbb{Z}\big[\sqrt{a}i\big]$
  or with integers in a quaternion algebra
  or with the integers in some other division algebra
- or a few other "exceptional" types

Consider them case-by-case. □

Example. Suppose $\Gamma = \mathrm{SO}(B;\mathbb{Z})$, where
$B(\overrightarrow{x}) = \sum_{i,j} a_{ij}x_i x_j$ with $a_{i,j} \in \mathbb{Z}$ and $a_{ij} = a_{ji}$.
We have

$$1 = \mathbb{Q}\text{-}\operatorname{rank}\Gamma = \text{max dim of totally null } \mathbb{Q}\text{-subspace}$$

so (after a change of basis),

$$B(\overrightarrow{x}) = x_1x_2 + a_3x_3^2 + a_4x_4^2 + \cdots + a_nx_n^2.$$

By normalizing, we may assume $a_3 = 1$.

- $\mathbb{R}$-$\operatorname{rank} G \geq 2 \implies$ wolog $a_4 > 0$.
- $\mathbb{Q}$-$\operatorname{rank}\Gamma = 1 \implies a_4$ *not* a perfect square.

Now

$$\Gamma \supset \mathrm{SO}(x_1x_2 - x_3^2 + a_4x_4^2) \doteq \mathrm{SL}\big(2, \mathbb{Z}[\sqrt{a}]\big).$$

Lemma (used above). $\mathrm{SO}(x_1x_2 - x_3^2 + a_4x_4^2) \doteq \mathrm{SL}\big(2, \mathbb{Z}[\sqrt{a}]\big)$.

Proof. Let

$$\begin{aligned} V &= \{\, \text{Hermitian mats in } \mathrm{Mat}_{2\times 2}\big(\mathbb{Q}(\sqrt{a})\big) \,\} \\ &= \left\{ \begin{bmatrix} x & y+\sqrt{a}z \\ y-\sqrt{a}z & w \end{bmatrix} \;\middle|\; x,y,z,w \in \mathbb{Q} \right\}. \end{aligned}$$

This is a 4-dimensional vector space over $\mathbb{Q}$.

$\mathrm{SL}\big(2, \mathbb{Z}[\sqrt{a}]\big)$ acts on V by $H \mapsto AH\overline{A}^T$,
and this action preserves the determinant

$$xw - (y+\sqrt{a}z)(y-\sqrt{a}z) = xw - (y^2 - az^2) = x2 - y^2 + az^2,$$

so this provides an embedding

$$\mathrm{SL}\big(2, \mathbb{Z}[\sqrt{a}]\big) \hookrightarrow \mathrm{SO}(x_1x_2 - x_3^2 + a_4x_4^2; \mathbb{Z}).$$

(In fact, this is an isomorphism, modulo finite groups.) □

Reference

Vladimir Chernousov, Lucy Lifschitz, and Dave Witte Morris:
Almost-minimal nonuniform lattices of higher rank,
Michigan Mathematical Journal (to appear).
http://arxiv.org/abs/0705.4330

Appendix

- $\mathrm{SL}(3,\mathbb{R})/\,\mathrm{SO}(3) \cong \left\{ \begin{matrix} 3\times 3 \text{ positive-definite, symmetric} \\ \text{matrices of determinant } 1 \end{matrix} \right\}.$
- $\mathrm{SL}(3,\mathbb{C})/\,\mathrm{SU}(3) \cong \left\{ \begin{matrix} 3\times 3 \text{ positive-definite, Hermitian} \\ \text{matrices of determinant } 1 \end{matrix} \right\}.$
- $\mathrm{Sp}(4,\mathbb{R})/\,\mathrm{SU}(2) \cong \left\{ \begin{matrix} 2\times 2 \text{ symmetric matrices over } \mathbb{C}, \\ \text{with determinant } 1, \\ \text{and positive-definite imaginary part} \end{matrix} \right\}$

  $= \text{“Siegel upper half space of genus 2.”}$

Remark. $\mathrm{Sp}(4,\mathbb{R}) = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \;\middle|\; \begin{bmatrix} A^TB = B^TA \\ C^TD = D^TC \\ A^TD - B^TC = I \end{bmatrix} \right\}$ acts by

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \circ M = (AM+C)(BM+D)^{-1}.$$

Notes typed and corrected February 12, 2008