What is a superrigid subgroup?

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Abstract

It is not difficult to see that every group homomorphism from $\mathbb{Z}^k$ to $\mathbb{R}^n$ extends to a homomorphism from $\mathbb{R}^k$ to $\mathbb{R}^n$. (Essentially, this is the fact that a linear transformation can be defined to have any desired action on a basis.) We will see other examples of discrete subgroups $\Gamma$ of connected groups $G$, such that the homomorphisms defined on $\Gamma$ can ("almost") be extended to homomorphisms defined on all of $G$. 
Eg. Group homomorphism $\phi: \mathbb{Z} \rightarrow \mathbb{R}^d$
(i.e., $\phi(m + n) = \phi(m) + \phi(n)$)

$\Rightarrow \phi$ extends to homo $\hat{\phi}: \mathbb{R} \rightarrow \mathbb{R}^d$.

Namely, define $\hat{\phi}(x) = x \cdot \phi(1)$.

Check:
- $\hat{\phi}(x + y) = \hat{\phi}(x) + \hat{\phi}(y)$
- $\hat{\phi}(n) = \phi(n)$
- $\hat{\phi}$ is continuous

(only allow continuous homos)

Eg. Group homomorphism $\phi: \mathbb{Z}^k \rightarrow \mathbb{R}^d$

$\Rightarrow \phi$ extends to homo $\hat{\phi}: \mathbb{R}^k \rightarrow \mathbb{R}^d$.

Proof. Standard basis $\{e_1, \ldots, e_k\}$ of $\mathbb{R}^k$.
Define $\hat{\phi}(x_1, \ldots, x_k) = \sum x_i \phi(e_i)$.

(“linear trans can do anything to a basis”)

Linear transformation $\Rightarrow$ homo of additive groups
Group Representation Theory:
study homos into Matrix Groups.

\[ \text{GL}_d(\mathbb{C}) = d \times d \text{ matrices over } \mathbb{C} \]
with nonzero determinant

This is a group under multiplication.

\[ \mathbb{R}^d \cong \begin{pmatrix}
1 & 0 & 0 & \mathbb{R} \\
0 & 1 & 0 & \mathbb{R} \\
0 & 0 & 1 & \mathbb{R} \\
0 & 0 & 0 & 1
\end{pmatrix} \]

So any homomorphism into \( \mathbb{R}^d \)
can be thought of as a homo into \( \text{GL}_{d+1}(\mathbb{C}) \).
Group homo $\phi: \mathbb{Z} \rightarrow \text{GL}_d(\mathbb{R})$

(i.e., $\phi(m + n) = \phi(m) \cdot \phi(n)$)

$\not\Rightarrow$ extends to homo $\hat{\phi}: \mathbb{R} \rightarrow \text{GL}_d(\mathbb{R})$.

(Only allow continuous homos.)

*Pf by contradiction.* Supse $\exists$ homo $\hat{\phi}: \mathbb{R} \rightarrow \text{GL}_d(\mathbb{R})$

with $\hat{\phi}(n) = \phi(n)$ for all $n \in \mathbb{Z}$.

$\hat{\phi}(0) = I \Rightarrow \det(\hat{\phi}(0)) = 1 > 0$

$\mathbb{R}$ connected

$\Rightarrow \hat{\phi}(\mathbb{R})$ connected

$\Rightarrow \det(\hat{\phi}(\mathbb{R}))$ connected

$\Rightarrow \det(\hat{\phi}(\mathbb{R})) > 0$

$\Rightarrow \det(\phi(1)) > 0$

Maybe $\det(\phi(1)) < 0.$

(Any $A \in \text{GL}_d(\mathbb{R})$, let $\phi(n) = A^n.$)
Group homo $\phi: \mathbb{Z} \to \text{GL}_d(\mathbb{R})$

$\not\Rightarrow$ extends to homo $\hat{\phi}: \mathbb{R} \to \text{GL}_d(\mathbb{R})$.

Because: maybe $\det(\phi(1)) < 0$.

$$\det(\phi(2)) = \det(\phi(1 + 1)) = \left(\det(\phi(1))\right)^2 > 0.$$  

In fact, $\det(\phi(\text{even})) > 0$.

May have to ignore odd numbers: restrict attention to even numbers.
Analogously, may need to restrict to multiples of 3 (or 4 or 5 or ...)

Restrict attention to multiples of $N$.

{multiples of $N$} is a subgroup of $\mathbb{Z}$

“Restrict attention to a finite-index subgroup”

**Prop.** Grp homo $\phi: \mathbb{Z}^k \rightarrow GL_d(\mathbb{R})$

$\Rightarrow \phi$ “almost” extends to homo $\hat{\phi}: \mathbb{R}^k \rightarrow GL_d(\mathbb{R})$

such that $\hat{\phi}(\mathbb{R}^k) \subset \overline{\phi(\mathbb{Z}^k)}$.

(“Zariski closure”)

This means $\mathbb{Z}^k$ is superrigid in $\mathbb{R}^k$.

“Homomorphisms defined on $\mathbb{Z}^k$ almost extend to be defined on $\mathbb{R}^k$”

Generalize to nonabelian groups.
**Lagrange interpolation:**

there is a polynomial curve

\[ y = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \]

through any \( n + 1 \) points.
Idea: Zar closure is like *convex hull*.

Image of $\phi$ controls image of $\hat{\phi}$.

_Eg._ If all matrices in $\phi(\mathbb{Z})$ commute, then all matrices in $\hat{\phi}(\mathbb{R})$ commute.

_Eg._ If all matrices in $\phi(\mathbb{Z})$ fix a vector $v$, then all matrices in $\hat{\phi}(\mathbb{R})$ fix $v$.

Good properties of $\phi(\mathbb{Z})$ carry over to $\hat{\phi}(\mathbb{R})$. 
\( \mathbb{Z}^k \) is a lattice in \( \mathbb{R}^k \). I.e.,

- \( \mathbb{R}^k \) is a (simply) connected matrix grp ("Lie group")
- \( \mathbb{Z}^k \) is a discrete subgroup
- all of \( \mathbb{R}^k \) is within a bounded distance of \( \mathbb{Z}^k \)
  \[ \exists C, \ \forall x \in \mathbb{R}^k, \ \exists m \in \mathbb{Z}^k, \ d(x, m) < C. \]

\( H \) is a lattice in \( G \)

**Cor.** Only countably many simply connected Lie groups have lattices.

Lie groups are of three types:

- solvable (many normal subgrps, e.g., abel)
- simple ("no" normal subgrps, e.g., \( \text{SL}_k(\mathbb{R}) \))
- combination (e.g., \( G = \mathbb{R}^k \times \text{SL}_k(\mathbb{R}) \))

More or less: \( H = \mathbb{Z}^k \times \text{SL}_k(\mathbb{Z}) \)

(\( H \) has solvable part and simple part)
Let us consider *solvable* groups.

A connected subgroup $G$ of $\text{GL}_d(\mathbb{C})$ is **solvable** if it is upper triangular

$$G \subset \begin{pmatrix} \mathbb{C}^* & \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{C}^* & \mathbb{C} \\ 0 & 0 & \mathbb{C}^* \end{pmatrix}$$

(or is after a change of basis).

*Eg.* All abelian groups are solvable.

**Proof.** Every matrix can be triangularized over $\mathbb{C}$. Pairwise commuting matrices can be simultaneously triangularized.
Examples of lattices.

\[ G = \begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} & \mathbb{R} \\ 0 & 1 & \mathbb{R} & \mathbb{R} \\ 0 & 0 & 1 & \mathbb{R} \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

\[ H = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 0 & 1 & \mathbb{Z} \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

\[ \overline{H} = G \quad \text{superrigid} \]

\[ G = \begin{pmatrix} \mathbb{R}^+ & 0 & 0 \\ 0 & \mathbb{R}^+ & 0 \\ 0 & 0 & \mathbb{R}^+ \end{pmatrix} \]

\[ H = \begin{pmatrix} 2^\mathbb{Z} & 0 & 0 \\ 0 & 2^\mathbb{Z} & 0 \\ 0 & 0 & 2^\mathbb{Z} \end{pmatrix} \]

\[ \overline{H} = G \quad \text{superrigid} \]
\[
G = \begin{pmatrix}
1 & \mathbb{R} & \mathbb{C} \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\quad H = \begin{pmatrix}
1 & \mathbb{Z} & \mathbb{Z} + \mathbb{Z}i \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\]

\[
\overline{G} = G 
\quad \overline{H} = G 
\]

\[
G' = \begin{pmatrix}
1 & t & \mathbb{C} \\
0 & 1 & 0 \\
0 & 0 & e^{2\pi it} \\
\end{pmatrix}
\quad \overline{G}' = \begin{pmatrix}
1 & \mathbb{R} & \mathbb{C} \\
0 & 1 & 0 \\
0 & 0 & T \\
\end{pmatrix}
\]

\[
H' = \begin{pmatrix}
1 & \mathbb{Z} & \mathbb{Z} + \mathbb{Z}i \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix} = H 
\]

*\(H\) is a lattice in both \(G\) and \(G'\).*

\[
\overline{H} = G \neq \overline{G}' \quad \text{so} \quad \overline{H} \neq \overline{G}'
\]

*\(H\) is not superrigid in \(G'\).*

E.g., the identity map \(\phi: H \to H\)
does not extend to homo \(\hat{\phi}: G' \to \overline{H}\).

*Proof.* \(\overline{H} = G\) is abelian, but \(G'\) is not abelian.
Prop. \( H \) superrigid in \( G \)
\[ \Rightarrow \overline{H} = \overline{G} \pmod{Z(G)}. \]

Converse:

Thm (Witte). A lattice \( H \) in a solvable grp \( G \) is superrigid iff \( \overline{H} = \overline{G} \pmod{Z(G)} \).

\( \overline{H} \neq \overline{G'} \): some of the rotations associated to \( G' \) do not come from rotations associated to \( H \)

\[
\begin{align*}
\text{rot} \left( \begin{array}{cc}
\alpha & * \\
0 & \beta \\
\end{array} \right) &= \left( \begin{array}{cc}
\frac{\alpha}{|\alpha|} & 0 \\
0 & \frac{\beta}{|\beta|} \\
\end{array} \right)
\end{align*}
\]
**Cor (Witte).** A lattice $H$ in a Lie group $G$ is “superrigid” iff

- $H = G \pmod{Z(G) \cdot \text{(cpct ss normal subgrp)}}$
- and simple part of $H$ is “superrigid.”

**Thm (Margulis Superrigidity Theorem).**
All lattices in $\text{SL}_n(\mathbb{R})$ are “superrigid” if $n \geq 3$. Similar for other simple Lie groups, $\mathbb{R}$-rank $\geq 2$.

**Cor (Margulis Arithmeticity Theorem).**
Every lattice in $\text{SL}_n(\mathbb{R})$ is “arithmetic” if $n \geq 3$. (like $\text{SL}_n(\mathbb{Z})$)

Only way to make a lattice: take integer points
(and minor modifications)
Similar for other simple groups with $\mathbb{R}$-rank $\geq 2$. 

Why superrigidity implies arithmeticity.

Let $\Gamma$ be a lattice in $\mathrm{SL}_n(\mathbb{R})$, and assume $\Gamma$ is superrigid.

We wish to show $\Gamma \subset \mathrm{SL}_n(\mathbb{Z})$, i.e., want every matrix entry to be an integer.

*First*, let us show they are algebraic numbers.

Suppose some $\gamma_{i,j}$ is transcendental.
Then $\exists$ field auto $\phi$ of $\mathbb{C}$ with $\phi(\gamma_{i,j}) = ???$

Define $\tilde{\phi}(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} \phi(a) & \phi(b) \\ \phi(c) & \phi(d) \end{pmatrix}$.

This map $\tilde{\phi}: \Gamma \to \mathrm{GL}_n(\mathbb{C})$ is a group homo.

Superrigidity: $\tilde{\phi}$ extends to $\hat{\phi}: \mathrm{SL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{C})$.

There are uncountably many different $\phi$’s, but $\mathrm{SL}_n(\mathbb{R})$ has only finitely many $n$-dim’l rep’ns (up to change of basis).
Γ is a superrigid lattice in $\text{SL}_n(\mathbb{R})$ and every matrix entry is an algebraic number.

Second, show matrix entries are rational.

Fact. Γ is generated by finitely many matrices. Matrix entries of these generators generate a finite-degree field extension of $\mathbb{Q}$.

“algebraic number field”

So $\Gamma \subset \text{SL}_n(F)$.

For simplicity, assume $\Gamma \subset \text{SL}_n(\mathbb{Q})$.

Third, show matrix entries have no denominators.

Actually, show denominators are bounded.

(Then finite-index subgrp has no denoms.)

Since Γ is generated by finitely many matrices, only finitely many primes appear in denoms.

So suffices to show each prime only occurs to bounded power.
Γ is a superrigid lattice in $\text{SL}_n(\mathbb{R})$
and every matrix entry is a rational number.

Need to show each prime only occurs to bounded power in denoms.

This is the conclusion of $p$-adic superrigidity:

**Thm** (Margulis).

*If $\Gamma$ is a lattice in $\text{SL}_n(\mathbb{R})$, $n \geq 3$, and $\phi: \Gamma \to \text{SL}_k(\mathbb{Q}_p)$ is a group homomorphism, then $\phi(\Gamma)$ has compact closure.*

*I.e., $\exists k$, no matrix in $\phi(\Gamma)$ has $p^k$ in denom.*

**Summary of proof:**

1) $\mathbb{R}$-superrigidity $\Rightarrow$ matrix entries “rational”

2) $\mathbb{Q}_p$-superrigidity $\Rightarrow$ matrix entries $\in \mathbb{Z}$