Ratner's Theorems
on Unipotent Flows

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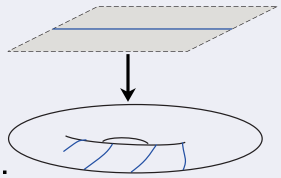
Ohio State U, June 1, 2010

Elementary example



Let M = torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$
- covering map $f\colon \mathbb{R}^2 \to M$
- L = line in $\mathbb{R}^2$.

If the slope of L is irrational,
it is classical that $f(L)$ is dense.

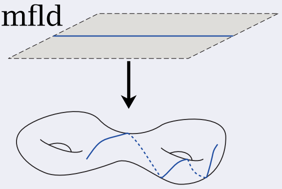
Exercise. Let M = n-torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$
- covering map $f\colon \mathbb{R}^n \to M$
- L = vector subspace of $\mathbb{R}^n$.

Closure $\overline{f(L)} = f(S)$ is a torus $\mathbb{T}^k$ ($\exists$ subspace S of $\mathbb{R}^n$).

The closure of $f(L)$ is a very nice submanifold of M.

Example in Riemannian geometry

Example



Let M = compact, hyperbolic n-mfld
- covering map $f\colon \mathbb{H}^n \to M$
- line $\mathbb{H}^1 \hookrightarrow \mathbb{H}^n$.

Closure $\overline{f(\mathbb{H}^1)}$ can be a fractal.

Consequence of Ratner's Thm

$\mathbb{H}^2 \subset \mathbb{H}^n \implies \overline{f(\mathbb{H}^2)} = f(\mathbb{H}^k)$ is a submfld of M
(immersed, maybe not embedded).

(Similar for other locally symmetric spaces.)

Recall

- $f\colon \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$
- L = vector subspace of $\mathbb{R}^n$

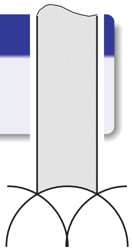
$\implies \overline{f(L)} = f(S)$, $\exists$ vector subspace S of $\mathbb{R}^n$.

- $\mathbb{R}^n$ is a Lie group.
- The subgroup $\mathbb{Z}^n$ is a lattice.

Generalization (Ratner's Theorem)

Replace:
- $\mathbb{R}^n$ with any Lie group G
- $\mathbb{Z}^n$ with any lattice Γ in G
- L with any subgroup of G
  that is generated by "unipotent" elements
- S with a closed subgroup of G

Ideas in proof of Ratner's Theorem



Example

$G = \mathrm{SL}(2,\mathbb{R}) = \{\ 2\times 2$ real mat's of det $1\ \}$.
Let $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. Then Γ is a lattice in G.

Other choices of Γ can make G/Γ compact.

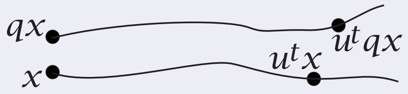
Definition

Define $u^t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ and $a^t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$.

Each is a homomorphism from $\mathbb{R}$ to $\mathrm{SL}(2,\mathbb{R})$.
u^t is a **unipotent** one-parameter subgroup.

Polynomial divergence

$$u^t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad a^t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$$

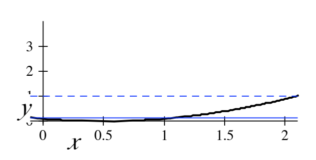


$d(x, qx) = ||q||$. $\qquad d(u^t x, u^t qx) = ||u^t q u^{-t}||$.

$$u^t q u^{-t} = \begin{bmatrix} \alpha + \gamma t & \beta + (\delta - \alpha)t - \gamma t^2 \\ \gamma & \delta - \gamma t \end{bmatrix}$$

Points move apart at polynomial speed. (*Slowly!*)

Points move apart at polynomial speed.



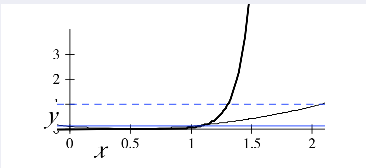
If 2 pts stay very close together for a length of time, then they stay close for just as long after that.

Corollary

In a unipotent flow:
if two points are very close together most of the time, then they are close together all of the time.

Contrast: $a^t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$ $\quad a^t q a^{-t} = \begin{bmatrix} \alpha & \beta e^{2t} \\ \gamma e^{-2t} & \delta \end{bmatrix}$

Points move apart at *exponential speed*.



Fact

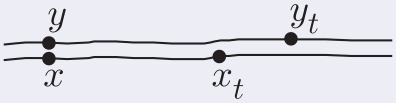
In a geodesic flow:
2 points can be close together 99.999% of the time, and wander independently the rest of the time.

Shearing

$$u^t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \; u^t q u^{-t} = \begin{bmatrix} \alpha + \gamma t & \beta + (\delta - \alpha)t - \gamma t^2 \\ \gamma & \delta - \gamma t \end{bmatrix}$$

Shearing

Fastest motion is parallel to the orbits.

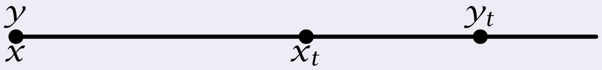


Corollary

If $x \approx y$, then $\exists t$, $y_t \approx x_{t+1}$.

Shearing

Fastest motion is parallel to the orbits.



Contrast: $a^t q a^{-t} = \begin{bmatrix} \alpha & \beta e^{2t} \\ \gamma e^{-2t} & \delta \end{bmatrix}$

Fastest motion is transverse to the orbits.

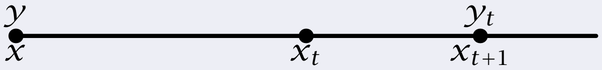
Theorem (Ratner)

$$\begin{tikzcd} SL(2,\mathbb{R})/\Gamma \arrow[r, "f"] \arrow[d, "u^t"] & Z \arrow[d, "\varphi_t"] \\ SL(2,\mathbb{R})/\Gamma \arrow[r, "f"] & Z \end{tikzcd}$$

(measurable, not constant a.e.) $\Longrightarrow$ $f^{-1}(z)$ *is countable* a.e.

Pf. Assume φ_1 has no fixed pts (so $d(\varphi_1(z), z) > \epsilon$) and f is uniformly continuous [Lusin's Thm].

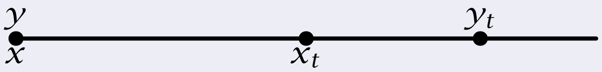
$f^{-1}(z)$ unctble $\Rightarrow$ $\exists\, x, y \in f^{-1}(z)$ with $x \approx y$.
Flow along the orbits until $d(x_t, y_t) = 1$.



Then $z_t = f(y_t) \approx f(x_{t+1}) = z_{t+1} = \varphi_1(z_t)$. $\rightarrow\leftarrow$

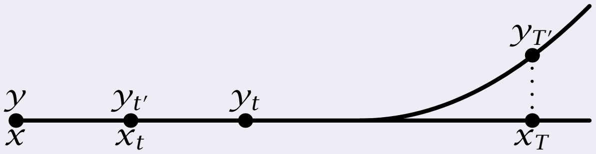
Shearing

Fastest motion is parallel to the orbits.



Key idea in Ratner's proof

Ignore motion along the orbit, and look at the *transverse* motion perpendicular to the orbit.



Example

$$u^t = \begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}, \; u^t q u^{-t} = \begin{bmatrix}\alpha+\gamma t & \beta + (\delta-\alpha)t - \gamma t^2 \\ \gamma & \delta - \gamma t\end{bmatrix}$$

Fastest motion is along $\{u^t\}$.

Ignoring this, largest terms are diagonal (in $\{a^t\}$)

Observation

$$a^t \begin{bmatrix}1 & *\\ 0 & 1\end{bmatrix} a^{-t} = \begin{bmatrix}1 & *\\ 0 & 1\end{bmatrix}: \quad a^t \textit{ normalizes } \{u^t\}.$$

Proposition

For action of a unipotent subgroup, the fastest transverse divergence is along the normalizer.

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Application

If C is a min'l closed, u^t-inv't subset of G/Γ,

(and C not contained in a single $N_G(\{u^t\})$-orbit)

then $a^s C = C$. (In fact, $C = G/\Gamma$.)

Proof.

Choose $x, y \in C$ with $x \approx y$ and $y \notin N_G(\{u^t\})x$.

Then $u^{t_1}x \approx a^s u^{t_2} y$.

Pretend $u^{t_1}x = a^s u^{t_2} y \in C \cap a^s C \neq \emptyset$.

But $u^t a^s C = a^s u^{t'} C = a^s C$, so $C \cap a^s C$ is u^t-inv't.

Minimality: $a^s C = C$. □

Ratner's Theorems

Ratner's Theorem on Orbit Closures

$\overline{\{u^t\}x} = Sx$, $\exists$ subgroup S of G.

Ratner's Equidistribution Theorem

$\{u^t\}x$ is *equidistributed* in Sx:

$$\frac{1}{T}\int_0^T f(u^t x)\,dt \to \int_{Sx} f\,d\mu \quad \text{for } f \in C_c(G/\Gamma)$$

where $\mu =$ (normalized) S-invariant volume form on Sx.

Ratner's Measure-Classification Theorem

Any ergodic u^t-invariant probability measure on G/Γ is (normalized) S-invariant volume form on Sx

for some subgroup S and some $x \in G/\Gamma$.

Further reading

chapter of my forthcoming book on arithmetic grps

- free PDF file on my web page
  http://people.uleth.ca/~dave.morris
  /books/IntroArithGroups.html

my book: *Ratner's Theorems on Unipotent Flows*

- free PDF file on my web page (or the arxiv)
  http://people.uleth.ca/~dave.morris
  /books/Ratner.html

Part 2:
A Ratner Theorem Proof

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Examples of Group Actions
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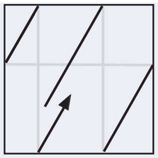
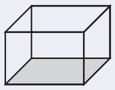
June 2, 2010

Examples



Let X = torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.
Any $v \in \mathbb{R}^2$ defines a flow on $\mathbb{T}^2$:
$\varphi_t(x) = x + tv$
If the slope of v is irrational,
it is classical that every orbit is dense (& unif dist).
Also: Lebesgue is the only inv't probability measure.



$v = (a, b, 0)$ defines flow φ_t on $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$
and $\mathbb{T}^2 \times \{0\}$ is invariant.
Lebesgue on $\mathbb{T}^2 \times \{0\}$ is an invariant measure.

$\forall v$, any ergodic invariant probability meas for φ_t on $\mathbb{T}^3$ is Lebesgue meas on some subtorus $\mathbb{T}^k$ $(0 \le k \le 3)$.

Generalization

Ratner's Measure-Classification Theorem

- G = Lie group
- Γ = lattice in G
- u^t = *unipotent* one-parameter subgrp of G
- μ = ergodic u^t-inv't probability measure on G/Γ

$\implies$ μ is (normalized) S-invariant volume form on Sx
for some closed subgroup S and some $x \in G/\Gamma$.

Today: $G = \mathrm{SL}(2,\mathbb{R})$.
Recall $u^t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ and $a^s = \begin{bmatrix} e^s & 0 \\ 0 & e^{-s} \end{bmatrix}$.

The volume on G/Γ is a finite, invariant measure.

Theorem (Furstenberg)

$\mathrm{SL}(2,\mathbb{R})/\Gamma$ *compact* $\Rightarrow$ *the only invariant probability measure for* u^t *is the ordinary volume.*

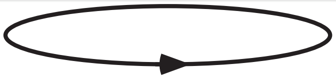
(In contrast, a^s has *many* invariant measures.)

Corollary

Every u^t*-orbit is uniformly distributed* (∴ *dense*).

Theorem (Dani)

$\mathrm{SL}(2,\mathbb{R})/\Gamma$ *noncompact* $\Rightarrow$ *other* ergodic u^t*-invariant measure is the measure supported on a closed orbit.*



1st step of the proof

Recall (from transverse shearing)

If C is a minimal compact, u^t-inv't subset of G/Γ,
(and C not contained in a single $N_G(\{u^t\})$-orbit)
then $a^s C = C$.

Similar argument:

$a^s \mu = \mu$ (unless μ is supported on a single $N_G(\{u^t\})$-orbit)

So we can use the dynamics of the a^s-flow:
entropy.

2nd step: entropy calculation

Theorem (from Anne's lecture)

- T = *vol-pres diffeo on manifold* M (cpct, smooth)
- *tangent bundle* $\mathcal{T}M = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_n$ (T-inv't),
  $\forall \xi \in \mathcal{E}_i$, $\|T(\xi)\| = \tau_i \|\xi\|$.

Then $h_{\mathrm{vol}}(T) = \sum_{\tau_i > 1} (\dim \mathcal{E}_i) \log \tau_i$.

Example (entropy of geodesic flow)

Recall $a^s q a^{-s} = \begin{bmatrix} \alpha & \beta e^{2s} \\ \gamma e^{-2s} & \delta \end{bmatrix}$.

$\mathcal{T}M = \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ * & 0 \end{bmatrix}$.

So $h_{\mathrm{vol}}(a^s) = 2|s|$.

Example. $h_{\mathrm{vol}}(a^s) = 2|s|$.

Proposition

Suppose ν is a^s-inv't.
Then $h_\nu(a^s) \le 2|s|$ with equality iff ν is u^t-inv't.

Corollary

μ is inv't under u^t and a^s $\implies$ $\mu =$ Lebesgue.

Proof.

μ is u^t-inv't $\implies h_\mu(a^s) = 2|s| \implies h_\mu(a^{-s}) = 2|s|$

$\implies \mu$ is invariant under $\begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix} = v^r$.

μ is invariant under $\langle u^t, a^s, v^r \rangle = \mathrm{SL}(2, \mathbb{R})$. □

Prove μ is a^s-invariant

We need to know a little bit about ergodicity.

Definition. Assume $\mu = \varphi_t$-inv't measure on X.
μ is *ergodic* for φ_t: $\nexists$ φ_t-inv't measurable set C
except $\mu(C) = 0$ or $\mu(C) = 1$.

Lemma

ergodic
$\iff$ *$\nexists$ nonconstant φ_t-invariant vector in $L^2(X;\mu)$*
$\iff$ *$\mu \ne \mu_1 + \mu_2$ unless μ_1 & μ_2 are scalar mults of μ*

Fact: { convex combs of ergodic measures }
is dense in { invariant measures }.

Remark

Howe-Moore Theorem implies
volume on G/Γ is ergodic for u^t
(or any other closed, noncpct subgroup of G).

If vol is ergodic for φ_t, then a.e. φ_t-orbit is dense.
In fact, the following fundamental theorem of Ergodic Theory implies they are unif dist:

Pointwise Ergodic Theorem

If μ is ergodic for φ_t, and $f \in L^1(\mu)$, then
$$\frac{1}{T}\int_0^T f(\varphi_t(x))\,dt \longrightarrow \int_X f\,d\mu \quad \text{for a.e. } x \in X.$$

Proof that μ is a^s-invariant

Let $\Omega = \{\, x \mid u^t\text{-orbit of } x \text{ is unif dist (w.r.t. } \mu)\,\}$
"generic set for μ"
Pointwise Ergodic Theorem: $\mu(\Omega) = 1$.
Since a^s normalizes u^t, we know

- $a^s\mu$ is u^t-inv't (and ergodic), and
- $a^s\Omega$ is the generic set for $a^s\mu$.

Choose $x, y \in \Omega$ with $x \approx y$. Then $u^{t_1}x \approx a^s\,u^{t_2}y$.
Pretend $z = u^{t_1}x = a^s\,u^{t_2}y \in \Omega \cap a^s\Omega$.
Orbit of z is equidistrib for both μ and $a^s\mu$, so $(\forall f)$
$$\int_{G/\Gamma} f\,d\mu = \lim \frac{1}{T}\int_0^T f(u^t z)\,dt = \int_{G/\Gamma} f\,d(a^s\mu),$$
so $\mu = a^s\mu$.

We should not pretend that $u^{t_1}x = a^s\,u^{t_2}y$.

Suppose $a^s\mu \ne \mu$. Then (as above) $\Omega \cap a^s\Omega = \varnothing$.
Same is true for nearby values of s
(because $\{\, s' \in \mathbb{R} \mid a^{s'}\mu = \mu \,\}$ is closed).
Let K be a compact subset of Ω with $\mu(K) > 1 - \epsilon$.

Then $d\left(K, \bigcup_{s' \approx s} a^{s'}K\right) > \delta > 0$.

Choose $x, y \in \Omega$ with $x \approx y$. Then $u^{t_1}x \approx a^{s'}\,u^{t_2}y$.
By Pointwise Ergodic Theorem (and polynomial divergence) we may assume $u^{t_1}x, u^{t_2}y \in K$.

This contradicts the fact that $d(K, a^{s'}K) > \delta$.