Some arithmetic groups that do not act on the circle

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Lecture 2: Proof for SL(2, Z/α) using bounded generation

Bounded generation by unip subgrps

Note: Invertible matrix \( \sim \) Id by row operations.

Key fact: \( g \in SL(2, \mathbb{Z}) \sim \) Id by integer (\( \mathbb{Z} \)) row ops.

Example:

\[
\begin{bmatrix} 13 & 31 \\ 5 & 12 \end{bmatrix} \sim \begin{bmatrix} 3 & 7 \\ 5 & 12 \end{bmatrix} \sim \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

\( \vdots \) \( U \) and \( V \) generate \( SL(2, \mathbb{Z}) \).

But # steps is not bounded: \( U \) and \( V \) do not boundedly generate \( SL(2, \mathbb{Z}) \).

Theorem (Liehl [1984])

\( SL(2, \mathbb{Z}[1/p]) \) boundedly gen’d by elem mats.

I.e., \( T \sim \) Id by \( \mathbb{Z}[1/p] \) col ops, # steps is bdd.

Proof.

\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix} a = a + kb, p \text{ is prim root}
\sim \begin{bmatrix} q & b \\ * & * \end{bmatrix} p^q \equiv b \pmod{q}; p^q = b + k'q
\sim \begin{bmatrix} q & p^q \\ * & * \end{bmatrix} \text{unit: can add anything to } q
\sim \begin{bmatrix} 1 & p^q \\ * & * \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ * & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

\( \Box \)

Theorem (Lifschitz-Morris [2004])

\( \Gamma = \text{large arithmetic group} \)
\( \equiv SL(3, \mathbb{Z}), SL(2, \mathbb{Z}[\alpha]) \), etc.
\( \alpha \equiv \text{irrational, algebraic, real} \)

Conjecture

\( \Gamma \) does not act on \( \mathbb{R} \).

(provably \( \sim \) no kernel)

\( \text{A faithful homomorphism } \phi: \Gamma \to \text{Homeo}^{+}(\mathbb{R}) \)

Proposition (Witte [1994])

\( \Gamma \) does not act on \( \mathbb{R} \) if \( \Gamma \equiv SL(3, \mathbb{Z}) \).

Theorem (Lifschitz-Morris [2004])

\( \Gamma \) does not act on \( \mathbb{R} \) if \( \Gamma \equiv SL(2, \mathbb{Z}[\alpha]) \).

Proof.

\( \sim \) I.e., \( SL \)

But \( \exists \Gamma \text{ faithful homomorphism } \phi: \Gamma \to \text{Homeo}^{+}(\mathbb{R}) \)

Key fact:

\( g \in SL(2, \mathbb{Z}) \sim \) Id by integer (\( \mathbb{Z} \)) row ops, but # steps is not bounded.

Remark: In \( SL(3, \mathbb{Z}) \), # steps is bounded [Carter-Keller, 1983].

Theorem (Carter-Keller-Paige, Lifschitz-Morris)

For \( \mathbb{Z}[\alpha] \) row ops, # steps is bounded.

\( \exists n, g \in SL(2, \mathbb{Z}[\alpha]), g = u_1 v_1 u_2 v_2 \cdots u_n v_n. \)

I.e., \( U \) and \( V \) boundedly generate \( \Gamma = SL(2, \mathbb{Z}[\alpha]) \).

So \( SL(2, \mathbb{Z}[\alpha]) = U V \Gamma \).

Theorem (Liehl [1984])

\( SL(2, \mathbb{Z}[1/p]) \) boundedly gen’d by elem mats.

I.e., \( T \sim \) Id by \( \mathbb{Z}[1/p] \) col ops, # steps is bdd.

Easy proof

Assume Artin’s Conjecture:

\( \forall r \neq \pm 1, \text{ perfect square.} \)

\( \exists \) \( \approx \) primes \( q, \text{ s.t. } r \equiv \text{ primitive root mod } q: \{ r, r^2, r^3, \ldots \} \mod q = \{ 1, 2, 3, \ldots, q-1 \} \)

Assume \( \exists q \in \) every arithmetic progression \( \{ a + kb \} \).

\( \exists q = a + kb, p \) is a primitive root modulo \( q \).

Theorem (Lifschitz-Morris [2004])

\( \Gamma = SL(2, \mathbb{Z}[1/p]) \) acts on \( \mathbb{R} \) \( \Rightarrow \) \( \Gamma \)-orbits are bdd.

Bdd generation: \( \Gamma = U V \Gamma \).

Bdd orbits: \( U \)-orbits and \( V \)-orbits are bounded.

Corollary

\( \phi: \Gamma \sim \text{Homeo}^{+}(\mathbb{R}) \Rightarrow \forall \Gamma \text{-orbit on } \mathbb{R} \) is bdd

\( \Rightarrow \) \( \Gamma \) has a fixed point.

Corollary

\( \Gamma \) cannot act on \( \mathbb{R} \).

Proof.

Spse \( \exists \) nontrivial action.

It has fixed points:

Remove them:

Take a connected component:

\( \Gamma \) acts on open interval \( \approx \mathbb{R} \) with no fixed pt.

Bounded orbits

Assume \( U \)-orbit and \( V \)-orbit of \( x \) not bdd above. Assume \( p \) fixes \( x \). (\( p \) does not have fixed pts, so not an issue.)

Wolog \( \overline{U}(x) < \psi(x) \).

Then \( \mathfrak{p}^n(\overline{U}(x)) < \mathfrak{p}^n(\psi(x)) \).

LHS = \( \mathfrak{p}^n(\overline{U}(x)) = (\mathfrak{p}^n\overline{U}(x)) \psi(x) \sim (\infty) \to \infty \).

RHS = \( \mathfrak{p}^n(\psi(x)) = (\mathfrak{p}^n\psi(x)) \psi(x) \to (x) < \infty \).

Assume \( \Box \).

\( \Box \)
Optional exercises

6) (harder) Assume \( \Gamma \) bdd gen (by cyclic subgrps).
Show \( \langle g^n \mid g \in \Gamma \rangle \) has finite index in \( \Gamma (\forall n \in \mathbb{Z}) \).

7) Assume:
- \( \Gamma_1 \) and \( \Gamma_2 \) are arith subgrps of \( G_1 \) and \( G_2 \), resp.
- \( G_1 \) and \( G_2 \) are simple Lie grps of higher real rank.
- \( \Gamma_1 \) is cocompact, but \( \Gamma_2 \) is not cocompact.

Use the Margulis Superrigidity Theorem to show \( \Gamma_2 \) is not isomorphic to a subgroup of \( \Gamma_1 \).

Further reading


