Some arithmetic groups that do not act on the circle

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Lecture 4
Intro to bounded cohomology
(used to prove actions have a fixed point)

Recall: group cohomology \( H^*(\Gamma; \mathbb{R}) \)
- *cochain* \( c: \Gamma^n \to \mathbb{R} \) is el’t of \( C^n(\Gamma; \mathbb{R}) \)
- *coboundary* \( \delta: C^n(\Gamma; \mathbb{R}) \to C^{n+1}(\Gamma; \mathbb{R}) \)
- \( H^n(\Gamma; \mathbb{R}) = \ker \delta_n / \text{Im} \delta_{n-1} \)

**Definition (bounded cohomology)**
\( H_b^n(\Gamma; \mathbb{R}) \): require all cochains to be *bdd* funcs on \( \Gamma \).

**Example**
- \( H^0(\Gamma; \mathbb{R}) = \mathbb{R} = H_0^0(\Gamma; \mathbb{R}) \).
- \( H^1(\Gamma; \mathbb{R}) = \{ \text{homomorphisms } c: \Gamma \to \mathbb{R} \} \)
- \( H^2_b(\Gamma; \mathbb{R}) = \{ \text{bounded} \ text{ homos } c: \Gamma \to \mathbb{R} \} = \{0\} \).

Interested in \( H^2_b(\Gamma) \) — applies to actions on the circle.

**Example**
Spse \( \Gamma \) acts on circle. I.e., \( \Gamma \subset \text{Homeo}_+(\mathbb{R}/\mathbb{Z}) \).
Each \( g \in \Gamma \) lifts to \( \hat{g} \in \text{Homeo}_+(\mathbb{R}) \).
Not unique: \( \hat{g}(t) = \hat{g}(t) + n \), \( \exists n \in \mathbb{Z} \).
Can choose \( \hat{g}(0) \in [0,1) \).
Let \( c(g, h) = \hat{g}(\hat{h}(0)) - \hat{gh}(0) \in \mathbb{Z} \).

**Exercise**
- \( c \) is a 2-cocycle:
  \[
  c(h, k) - c(gh, k) + c(h, gk) - c(g, h) = 0
  \]
- \( c(g, h) \in \{0, 1\} \).

So \( [c] \in H^2_b(\Gamma; \mathbb{Z}) \). The *bdd Euler class* of the action.
Well defined: independent of basepoint “0”, etc.

**Proposition (Ghys)**
\( [c] = 0 \text{ in } H^2_b(\Gamma; \mathbb{Z}) \iff \Gamma \text{ has a fixed point in } S^1 \).

**Proof.**
(\( \Rightarrow \)) Wolog fixed point is 0.
Then \( \hat{g}(0) = 0 \), so \( c(g, h) = 0 \) for all \( g, h \).
(\( \Leftarrow \)) \( c(g, h) = \varphi(gh) - \varphi(g) - \varphi(h), \exists \text{ bdd } \varphi: \Gamma \to \mathbb{Z} \).
Let \( \hat{g}(\hat{h}) = \hat{g}(\hat{h}) + \varphi(\hat{h}) \), so
- \( \hat{g} \hat{h} = gh \), so \( \hat{g} \) is a lift of \( \Gamma \) to \( \text{Homeo}_+(\mathbb{R}) \), and
- \( |\hat{g}(0)| \leq |\hat{g}(0)| + |\varphi(h)| \leq 1 + \|\varphi\| \).
\( \hat{g} \)’s orbit of 0 is bdd subset of \( \mathbb{R} \), so has a supremum,
which is fixed pt of \( \hat{g} \); image in \( S^1 \) is fixed pt of \( \Gamma \).

**Theorem (Burger-Monod)**
Comparison map \( H^2_b(\Gamma; \mathbb{R}) \to H^2(\Gamma; \mathbb{R}) \) is injective
if \( \Gamma \) is large arith group.

**Kernel of the comparison map**
Let \( c \in Z^2_b(\Gamma; \mathbb{R}) \). Assume \( [c] = 0 \text{ in } H^2(\Gamma; \mathbb{R}) \).
I.e., \( c = \delta \alpha, \exists \alpha \in C^1(\Gamma; \mathbb{R}) \).
\( |\alpha(gh) - \alpha(g) - \alpha(h)| = |\delta \alpha(g, h)| \leq \|c\|_\infty \) is bdd.
\( \alpha \) is almost a homo — a *quasimorphism*.

**Exercise**
Kernel of \( H^2_b(\Gamma) \to H^2(\Gamma) \) is \( \text{NearHom}(\Gamma; \mathbb{R}) \).
NearHom(\Gamma; \mathbb{R}) = \{ \alpha: \Gamma \to \mathbb{R} \mid \text{bdd dist from hom} \}
Exercise

Kernel of $H^2_b(F_2) \to H^2(\Gamma)$ is Quasimorphisms$(\Gamma, \mathbb{R})$.

Example: $H^2_b(F_2)$ is infinite-dimensional.

Proof. Construct lots of quasimorphisms (not homos).

Homo $\varphi_a(x) =$ the (signed) # occurrences of $a$ in $x$.

E.g., $\varphi(a^2 b a^3 b^{-3} a^{-7} b^2) = 2 + 3 - 7 = -2$.

Every homo $F_2 \to \mathbb{R}$ is a linear comb of $\varphi_a$ and $\varphi_b$.

$\varphi_{ab}(x) =$ # occurrences of $ab$ in $x$ (reduced)

E.g., $\varphi_{ab}(a^2 b a^3 b^{-3} a^{-7} b^2) = 1 - 1 = 0$.

Exercise: 1) $\varphi_w$ is a quasimorphism, $\forall$ reduced $w$.

2) $\varphi_w$ is not within bdd distance of lin span of

$\{\varphi_b, \varphi_a, \varphi_{ab}, \varphi_{ab}^k, \varphi_{ab}^{k+2}, \varphi_{ab}^{k+3}, \ldots\}$.

Example: homomorphism $\varphi : \Gamma \to \mathbb{R}$

$\Rightarrow \{g \in \Gamma \mid \varphi(g) > 0\}$ is normal semigroup.

Exercise: Spse $\varphi : \Gamma \to \mathbb{R}$ unbdd quasimorphism. Stabilize: let $\varphi(g) = \lim (\varphi(g^n/n)$.

$\Rightarrow \varphi$ is unbounded quasimorphism.

$\Rightarrow (\varphi(h^{-1}gh) = \varphi(g)$.

$\Rightarrow \{g \in \Gamma \mid \varphi(g) > C\}$ is normal semigroup.

Open Problem. For $\Gamma = \text{SL}(3,\mathbb{Z})$:

- Every normal semigroup in $\Gamma$ is a subgroup.
- $\forall g \in \Gamma$, $e$ is a product of conjugates of $g$.
- $\mathfrak{F}$ (nonempty) bi-invariant partial order on $\Gamma$.

All are equivalent. ($\$100$ for solution)

Further reading

- M. Gromov: Volume and bounded cohomology. 
  [http://archive.numdam.org/article/PMIHES_1982__56__5_0.pdf]


  [http://arxiv.org/abs/0811.0051]

  [http://dx.doi.org/10.1090/S0002-9939-98-04368-8]