How to make infinitely large numbers from two-player games

Dave Morris, 7 December 2011

Abstract. We will talk about certain strategy games, in which the moves alternate between two players. Chess, checkers, and Go are some of the games we could discuss, but, to keep things simple, we will stick to easier examples. John H. Conway discovered that analyzing who will win from a given starting position has some very interesting consequences. In particular, we will see how to add two games (or subtract them, or multiply them), and we will encounter numbers that are infinitely large. No advanced mathematical training will be needed to understand most of this talk, but it would be helpful to have heard of “Dedekind cuts”.

The main talk will be preceded by a short explanation of “Zero-Knowledge Proofs.” These allow you to convince someone you know how to prove a theorem, without giving them any information at all about the proof (except how long it is).

Hackenbush.

This game is played by two players:

- Black (a.k.a. Left, the good guys, “us,” “you”), and
- Red (a.k.a. Right, the bad guys, “them,” “me”).

The two players take turns erasing a single edge of their colour.

Edges disappear if they are not connected to the ground by a path.

You lose if it’s your turn and you have no legal move.

Eg. Suppose red goes first in the following game:

Play could proceed as follows (where we colour an edge gray if it is being erased on that move):

Since it is red’s turn, but he or she has no edges to erase, we win.

We can describe any game by listing the legal moves for each player.

(Each move results in a new position — i.e., a game.)

\[ G = \{ L \mid R \} \] means that \( L \) is the set of moves for Left and \( R \) is the set of moves for Right

Eg.
Defn. (J. H. Conway)
If \( L \) and \( R \) are any sets of games, then \( \{ L \mid R \} \) is a game.
(And all games are constructed in this way.)

Eg.
- At the start, we have no games, so \( L \) and \( R \) must be the empty set:
  \( \{ \mid \} = \emptyset = \) the empty game = 0.
- \( \{ 0 \mid \} = \frac{1}{1} = 1 \) (We are 1 move ahead.)
- \( \{ \mid 0 \} = \frac{1}{1} = -1 \) (We are 1 move behind.)
- \( \{ 1 \mid \} = \frac{1}{1} = 2 \) (We are 2 moves ahead.)
- \( \{ \mid -1 \} = \frac{1}{1} = -2 \) (We are 2 moves behind.)
- \( \{ 0 \mid 1 \} = \frac{1}{1} = 1/2 \) (We are half a move ahead.)
  To understand what this means, we need some theory.
  Assump. Today, we assume \( L < R \):
  Every element of \( L \) is less than every element of \( R \) — like in a “Dedekind cut”.

Note that the YA game breaks up into two completely separate subgames:
a move in the “Y” has no effect on the “A”, and vice-versa.
In this situation, we write:

\[
\begin{array}{cc}
\text{Y} & \text{A} \\
\text{Y} & + \\
\end{array}
\]

In \( G + H \), a player can either
- make a legal move in \( G \) (and leave \( H \) untouched),
or
- make a legal move in \( H \) (and leave \( G \) untouched).

Formally if \( G = \{ \ell_i \mid r_j \} \) and \( G' = \{ \ell'_i \mid r'_j \} \), then
\( G + G' = \{ \ell_i + G', G + \ell'_i \mid r_j + G', G + G'_j \} \)

The game \(-G\) is the opposite of game \( G\):
change Red edges to Black, and vice-versa.

Eg.

\[
\begin{array}{cc}
\text{-Y} & \text{Y} \\
\text{-Y} & = \\
\end{array}
\]

Defn.
- \(-G = \{ -R \mid -L \} \).
- \( G = 0 \) iff 2nd player can always win
  (by playing well).
- \( G = H \) iff \( G - H = 0 \).
Prop. $G - G = 0$.

Proof. We wish to show that the 2nd player has a winning strategy:
Whenever the first player makes a move, the 2nd player can make the corresponding move in the other copy of $G$.

□

Eg. Play a game of $A - A$ against a volunteer

A similar argument shows that the above definitions are consistent:

Exer. If $G_1 = G_2$ and $H_1 = H_2$, then $G_1 + H_1 = G_2 + H_2$.

[Hint: $(G_2 + H_2) - (G_1 + H_1) = (G_2 - G_1) + (H_2 - H_1) = 0 + 0 = 0$.]

□

Exer. Verify that $\frac{1}{2} = \frac{1}{2}$.

Proof. We show $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 0$. (2nd player wins)

If Black goes first, he must move to $\frac{1}{2}$. Then Red can move to $\frac{1}{2}$.
Since Black needs to move first at this point, he will lose.

If Red goes first, she can move to $\frac{1}{2}$ or $\frac{1}{2}$.
Then Black can move to $\frac{1}{2}$ or $\frac{1}{2}$.
Then Red must move to $\frac{1}{2}$.
So Black moves to $\frac{1}{2}$ and Red loses, since she has no move.

□

Exer. Show $\sqrt[3]{A} = 1\frac{1}{4}$.

Thm. $\{L \mid R\} = x$ iff $x$ is the simplest number such that $L < x < R$.

Eg.

- $\{2\} = 3 = \{0,1,2\}$
- $\{2|3\} = \frac{2}{3}$
- $\{2|2\frac{1}{2}\} = \frac{2}{3}$
  (powers of 2 are the simplest denominators)

Eg.

- $\omega := \{0,1,2,\ldots\}$ is an infinitely large number.
- $\{0 \mid 1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots\} = \frac{1}{\omega}$ is infinitesimal
  (infinitely small).
- $\{\omega \mid \} = \omega + 1$.
- $\{1,2,3,\ldots \mid \omega\} = \omega - 1$.
- $\{1,2,3,\ldots \mid \omega,\omega - 1,\omega - 2,\ldots\} = \frac{\omega}{2}$.
- $\{1,2,3,\ldots \mid \omega,\omega/2,\omega/3,\omega/4,\ldots\} = \sqrt{\omega}$.
- $\{1,2,3,\ldots \mid \omega,\omega^2,\omega^3,\omega^4,\ldots\} = e^{\omega}$.
- $\{1,2,3,\ldots \mid \omega,\sqrt{\omega},\sqrt[3]{\omega},\sqrt[4]{\omega},\ldots\} = \log \omega$. 
Eg. We verify that \( \{1, 2, 3, \ldots \mid \omega, \omega - 1, \omega - 2, \ldots \} = \omega/2 \).

**Proof.** The game \( \frac{\omega}{2} + \frac{\omega}{2} - \omega \) should be a 2nd-player win.

If Left goes first, he must move to \( n + \frac{\omega}{2} - \omega \).
Then Right can move to \( n + (\omega - n) - \omega = 0 \), so Right wins.

If Right goes first, she can move to either \( (\omega - n) + \frac{\omega}{2} - \omega \)
or \( \frac{\omega}{2} + \frac{\omega}{2} - n \).
Then Left can move to \( (\omega - n) + n - \omega = 0 \) or \( \frac{\omega}{2} + n - n \).
Assume the latter. Then Right must move to \( (\omega - m) + (n - n) \),
so Left can move to \( (m - m) + (n - n) = 0 \), so Left wins. \( \Box \)

Rem. The numbers defined in this way are called **surreal numbers**.
They form a (real-closed) field:

- you can add, subtract, multiply, and divide.
Every polynomial of odd degree has a root.
- but problems remain in understanding logs, exponentials, integration, etc.

**References**