

**The Margulis Normal Subgroups Theorem**  
**Lecture 3: Proof of the ‘Black Box’ Result**

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**Black Box.**

- $\Gamma = \mathrm{SL}(3, \mathbb{Z})$ ,
  - $G = \mathrm{SL}(3, \mathbb{R})$ ,
  - $P = \begin{bmatrix} * & * & * \\ & * & * \\ & & * \end{bmatrix} \subset G$
  - $Z = \text{compact, metrizable space with an action of } \Gamma$
  - $\psi: G/P \rightarrow Z$  ( $\Gamma$ -equivariant)
- $\implies$  action of  $\Gamma$  on  $Z$  extends to (measurable) action of  $G$  (a.e.)  
 s.t.  $\psi$  is (essentially)  $G$ -equivariant.

*Remark.* A more concrete (but equivalent) way of stating the conclusion of the lemma is:

- $\exists$   $\Gamma$ -equivariant measurable  $\phi: Z \cong G/Q$  (a.e.),  
 for some closed subgroup  $Q$  of  $G$  that contains  $P$ .

The space  $Z$  is not known explicitly. Rather than trying to understand it directly, we will study  $Z$  by looking at the algebra of functions on it. (This is the usual practice in modern algebraic geometry.) In order to deal with the measurable map  $\psi$ , it is natural to consider a class of measurable functions. Thus, we note that  $\psi$  induces a  $\Gamma$ -equivariant inclusion

$$\psi^*: \mathcal{L}^\infty(Z) \hookrightarrow \mathcal{L}^\infty(G/P).$$

Furthermore, it is simpler to look at only the subset consisting of  $\{0, 1\}$ -valued functions (the characteristic functions of sets). We can identify  $\mathcal{B}(Z)$  with a subset of  $\mathcal{B}(G/P)$ :

$$\mathcal{B}(Z) \subset \mathcal{B}(G/P).$$

Now  $\mathcal{B}(Z)$  is a  $\Gamma$ -invariant Boolean subalgebra that is weak\*-closed (which implies that it is closed under convergence in measure), and we wish to show that it is  $G$ -invariant.

The Black Box can be reformulated in these terms:

**Black Box’.** If  $\mathcal{B}$  is any  $\Gamma$ -invariant, weak\*-closed Boolean subalgebra of  $\mathcal{B}(G/P)$ , then  $\mathcal{B}$  is  $G$ -invariant.

To make things easier, let us settle for a lesser goal temporarily:

**Lesser goal.** If  $\mathcal{B}$  is any nontrivial,  $\Gamma$ -invariant, weak\*-closed Boolean subalgebra of  $\mathcal{B}(G/P)$ , then  $\mathcal{B}$  contains a nontrivial  $G$ -invariant Boolean algebra.

*Definition.* “Nontrivial” means that the Boolean algebra is not just  $\{0, 1\}$  (constant functions).

*Remark.*

- To achieve the lesser goal, we will find some  $\bar{f} \in \mathcal{B}(G/P) \setminus \{0, 1\}$ , such that  $G\bar{f} \subset \mathcal{B}$ .
- The proof of Black Box’ is similar: let  $\mathcal{B}_G$  be the (unique) maximal  $G$ -invariant subalgebra of  $\mathcal{B}$ . If  $\mathcal{B}_G \neq \mathcal{B}$ , we will find some  $\bar{f} \in \mathcal{B}(G/P) \setminus \mathcal{B}_G$ , such that  $G\bar{f} \subset \mathcal{B}$ .

*Assumption.* To simplify the algebra, let us replace  $\mathrm{SL}(3, \mathbb{R})$  with  $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ .

*Notation.*

- $G = G_1 \times G_2$ , where  $G_1 = G_2 = \mathrm{SL}(2, \mathbb{R})$ ,
- $P = P_1 \times P_2$ , where  $P_i = \begin{bmatrix} * & * \\ & * \end{bmatrix} \subset G_i$ ,
- $V = V_1 \times V_2$ , where  $V_i = \begin{bmatrix} 1 & \\ * & 1 \end{bmatrix} \subset G_i$ ,
- $U = U_1 \times U_2$ , where  $U_i = \begin{bmatrix} 1 & * \\ & 1 \end{bmatrix} \subset P_i$ ,
- $\Gamma =$  irreducible lattice in  $G$ ;  
 i.e.,  $\Gamma$  is discrete,  $G/\Gamma$  has finite volume,  
 and  $\Gamma$  projects densely into  $G_1$  and  $G_2$   
 (e.g.,  $\Gamma = \mathrm{SL}(2, \mathbb{Z}[\sqrt{2}])$ ),
- $Z =$  compact, metrizable space with an action of  $\Gamma$ ,
- $\mathcal{B} =$  closed,  $\Gamma$ -invariant subalgebra of  $\mathcal{B}(G/P)$ .

*Remark.* Here are two descriptions of  $G/P = (G_1/P_1) \times (G_2/P_2)$ :

- $G/P \cong V_1 \times V_2$  (a.e.)

- $G/P = \mathbb{R}P^1 \times \mathbb{R}P^1 \cong \mathbb{R}^2$  (a.e.).

Note that, for the action of  $G_1$  on  $\mathbb{R}^2$ , we have

- $\begin{bmatrix} k^{-1} & \\ & k \end{bmatrix} (x, y) = (k^2x, y)$ , and
- $\begin{bmatrix} 1 & \\ t & 1 \end{bmatrix} (x, y) = (x + t, y)$ .

The proof of the lesser goal employs two preliminary results.

**Lemma (Lebesgue Differentiation Theorem).**

Let  $f \in \mathcal{L}^1(\mathbb{R}^n)$ . For a.e.  $p \in \mathbb{R}^n$ , we have

$$\lim_{r \rightarrow 0} \frac{1}{\mu(B_r(0))} \int_{B_r(p)} f d\mu = f(p).$$

Equivalently,

$$\lim_{k \rightarrow \infty} \frac{1}{\mu(B_1(0))} \int_{B_1(0)} f \left( p + \frac{x}{k} \right) d\mu(x) = f(p).$$

Applying Fubini's Theorem yields:

**Corollary.** Let

- $f \in \mathcal{L}^\infty(\mathbb{R}^2)$ ,
- $a = \begin{bmatrix} k^{-1} & \\ & k \end{bmatrix} \in G_1$ , for some  $k > 1$ , and
- $\pi: \mathbb{R}^2 \rightarrow \{0\} \times \mathbb{R}$  be the projection onto the  $y$ -axis.

Then, for a.e.  $v \in V_1$ ,

$$a^n v f \text{ converges in measure to } \overline{v f} = (v f) \circ \pi.$$

The other result we need is also a dynamical statement:

**Proposition.** For a.e.  $v \in V_1$ ,  $\Gamma v a^{-\mathbb{N}}$  is dense in  $G$ .

*Proof.* We wish to show  $a^{\mathbb{N}} v^{-1} \Gamma = G$ ;

i.e., the (forward)  $a$ -orbit of  $v^{-1}$  is dense in  $G/\Gamma$ .

Let  $\mathcal{O}$  be an open subset of  $G/\Gamma$ , and  $E = \cup_{n < 0} a^n \mathcal{O}$ .

Clearly,  $a^{-1}E \subset E$ . Since  $\mu(a^{-1}E) = \mu(E)$ ,  $E$  is  $a$ -invariant (a.e.).

Now, to show  $E = G/\Gamma$  (a.e.), it suffices to show  $a$  is ergodic on  $G/\Gamma$ :

there are no nontrivial  $a$ -invariant measurable subsets.

Let  $f$  be the characteristic function of  $E$ ; then  $f$  is  $a$ -invariant.

For  $v \in V_1$ , we have  $a^{-n} v a^n \rightarrow e$ , so

$$v f = v a^n f = a^n (a^{-n} v a^n) f \doteq a^n e f = a^n f = f,$$

so  $f$  is  $V_1$ -invariant. Similarly (replace  $a$  with  $a^{-1}$ ),  $f$  is  $U_1$ -invariant.

Therefore,  $f$  is left-invariant under  $\langle U_1, V_1 \rangle = G_1$ .

Since  $G_1 \triangleleft G$ , this implies  $f$  is right-invariant under  $G_1 \Gamma$ .

Since  $G_1 \Gamma$  is dense in  $G$ , we see that  $f$  is  $G$ -invariant; hence, constant.

We have shown, for a.e.  $g \in G$ , that  $\Gamma g a^{-\mathbb{N}}$  is dense, but we wish to know this for a.e. point in the submanifold  $V_1$ . We leave this as an exercise. [Hint: Show that if  $\Gamma g a^{-\mathbb{N}}$  is dense, then the same is true when  $g$  is replaced by any element of  $g U_1 C_G(a)$ .]  $\square$

**Proof of the lesser goal.**

Since  $\mathcal{B}$  is nontrivial, it contains some nonconstant  $f$ .

Now  $f$  cannot be essentially constant on almost every vertical and horizontal line, so we may assume

$$\{v \in \mathbb{R}^2 \mid f \text{ is not ess constant on the vertical line thru } v\}$$

has positive measure.

The two preliminary results tell us we may choose a  $v$  in this set, with the additional properties that

- $a^n v f \rightarrow \overline{v f}$ , and
- $\Gamma v^{-1} a^{-\mathbb{N}}$  is dense in  $G$ .

Now, for any  $g \in G$ , there exist  $\gamma_i \in \Gamma$  and  $n_i \rightarrow \infty$ , such that

$$g_i := \gamma_i v^{-1} a^{-n_i} \rightarrow g.$$

Then we have  $g_i a^{n_i} v = \gamma_i \in \Gamma$ , so the  $\Gamma$ -invariance of  $\mathcal{B}$  implies

$$\mathcal{B} \ni \gamma_i f = g_i a^{n_i} v f \rightarrow g \overline{v f}.$$

Since  $\mathcal{B}$  is closed, we conclude that  $g \overline{v f} \in \mathcal{B}$ .

Since  $g$  is an arbitrary element of  $G$ , this means  $G \overline{v f} \subset \mathcal{B}$ .

From the choice of  $v$ , we know that  $\overline{v f}$  is not essentially constant.  $\square$

**Proof of Black Box' (for  $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ ).**

Let  $\mathcal{B}_G$  be the largest  $G$ -invariant subalgebra of  $\mathcal{B}$ .

Since  $P$  is the stabilizer of a point in  $\mathbb{R}P^1 \times \mathbb{R}P^1$ , and has only 4 orbits, it is easy to see that the only subgroups of  $G$  containing  $P$  are

$$P, G_1 \times P_2, P_1 \times G_2, \text{ and } G.$$

It follows that the only  $G$ -invariant subalgebras of  $\mathcal{B}(G/P)$  (and, hence, the only possibilities for  $\mathcal{B}_G$ ) are

- $\mathcal{B}(\mathbb{R}^2)$ ,
- { functions constant on horizontal lines },
- { functions constant on vertical lines }, and
- $\{0, 1\}$ .

Suppose  $\mathcal{B} \neq \mathcal{B}_G$ . (This will lead to a contradiction.) There is some  $f \in \mathcal{B}$ , such that  $f \notin \mathcal{B}_G$ . Because  $\mathcal{B} \neq \{0, 1\}$ , the Lesser Goal tells us that  $\mathcal{B}_G \neq \{0, 1\}$ . Also, we know  $\mathcal{B}_G \neq \mathcal{B}(\mathbb{R}^2)$  (otherwise  $\mathcal{B} = \mathcal{B}_G$ ). Thus, we may assume, by symmetry, that

$$\mathcal{B}_G = \{ \text{functions constant on vertical lines} \}.$$

So  $f$  is *not* essentially constant on vertical lines.

Applying the proof of the Lesser Goal yields  $\overline{vf}$ , such that

- $\overline{vf}$  is *not* essentially constant on vertical lines, and
- $G\overline{vf} \subset \mathcal{B}$ , so  $\overline{vf} \in \mathcal{B}_G$ .

This contradicts our characterization of  $\mathcal{B}_G$ .  $\square$

### Modifications for $\text{SL}(3, \mathbb{R})$

- Let  $V = \begin{bmatrix} 1 & & \\ * & 1 & \\ * & * & 1 \end{bmatrix}$ ,  $V_1 = \begin{bmatrix} 1 & & \\ * & 1 & \\ & & 1 \end{bmatrix}$ ,  $V_2 = \begin{bmatrix} 1 & & \\ & 1 & \\ * & & 1 \end{bmatrix}$ .

We have  $V = \langle V_1, V_2 \rangle$ .

- Just as was the case for  $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ , there are exactly 4 subgroups containing  $P$ . Thus, there are only 4  $G$ -invariant subalgebras of  $\mathcal{B}(G/P)$ . The subgroups are  $P$ ,  $G$ ,

$$\begin{bmatrix} * & * & * \\ * & * & * \\ & & * \end{bmatrix} = \langle V_1, P \rangle, \quad \text{and} \quad \begin{bmatrix} * & * & * \\ & * & * \\ & * & * \end{bmatrix} = \langle V_2, P \rangle.$$

Hence, identifying  $\mathcal{B}(G/P)$  with  $\mathcal{B}(V)$ , the  $G$ -invariant subalgebras of  $\mathcal{B}(V)$  are

- $\mathcal{B}(V)$ ,
- $\{0, 1\}$ ,
- right  $V_1$ -invariant functions, and
- right  $V_2$ -invariant functions.

*Remark.* (The homogeneous spaces corresponding to the last two subgroups are  $\mathbb{R}P^2$  and the Grassmannian  $\mathbf{G}_{2,3}$  of 2-planes

in  $\mathbb{R}^3$ . Hence, in geometric terms, the  $G$ -invariant Boolean subalgebras are  $\mathcal{B}(G/P)$ ,  $\{0, 1\}$ ,  $\mathcal{B}(\mathbb{R}P^2)$ , and  $\mathbf{G}_{2,3}$ .

- Let  $\pi$  be the projection onto  $V_2$  in the semidirect product

$$V = V_2 \ltimes V_2^\perp, \quad \text{where } V_2^\perp = \begin{bmatrix} 1 & & \\ * & 1 & \\ * & & 1 \end{bmatrix}.$$

- For  $a = \begin{bmatrix} 1/k^2 & & \\ & k & \\ & & k \end{bmatrix} \in G$ , we have

$$a \begin{bmatrix} 1 & & \\ x & 1 & \\ z & y & 1 \end{bmatrix} P = \begin{bmatrix} 1 & & \\ k^3 x & 1 & \\ k^3 z & y & 1 \end{bmatrix} P.$$

Therefore, a generalization of the Lebesgue Differentiation Theorem tells us, for  $f \in \mathcal{B}(G/P) = \mathcal{B}(V)$  and a.e.  $v \in V_2^\perp$ , that  $a^n v f$  converges in measure to  $\overline{vf} = (vf) \circ \pi$ .

### References

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(Black Box' is stated in Theorem 4.2.11, and the Black Box is Corollary 2.13.)

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(The Black Box and Black Box' are stated in Theorems 8.1.3 and 8.1.4. See Sections 8.2 and 8.3 for the proofs.)