Using left-invariant orders to study actions on 1-manifolds

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Lecture 3

- Theorem of Ghys and Burger-Monod
- Kazhdan's property (T) and theorem of Navas

Theorem (Ghys, Burger-Monod)

Let $\Gamma =$ *subgroup of finite index in* $\mathrm{SL}(3,\mathbb{Z})$ *or lattice in* $\mathrm{SL}(3,\mathbb{R})$.
If $\Gamma \hookrightarrow \mathrm{Homeo}_+(S^1)$, *then* $\exists$ *finite orbit.*

Proof of Ghys

- *amenability* (Poisson bdry, Furstenberg bdry)
- *ergodic theory* (measurable theory of grp actions)

Exercise

$\exists$ Γ-*inv't prob meas on* $S^1 \implies \exists$ *finite orbit.*

Hint: Assume abelianization of Γ *is finite.*

Proof of Ghys

$\exists$ Γ-invariant probability measure on S^1.

Assume $\Gamma \hookrightarrow \mathrm{Homeo}_+(S^1)$, $\Gamma = \mathrm{SL}(3,\mathbb{Z})$ (latt in $\mathrm{SL}(3,\mathbb{R})$)

Key fact (amenability)

$$\mathbf{F} = \begin{matrix}\text{Furstenberg}\\ \text{boundary}\end{matrix} = \begin{matrix}\text{flag}\\ \text{variety}\end{matrix} = \{(\ell,\Pi) \mid \ell \subset \Pi \subset \mathbb{R}^3\}$$

$\Longrightarrow$ $\exists$ Γ-equivariant, *random* map $\mathbf{F} \to S^1$.

$\psi \colon \mathbf{F} \to \mathrm{Prob}(S^1)$ Γ-equivariant, meas'ble.

Proof of Ghys: ψ *constant.* (Then $\psi(\Gamma)$ is Γ-inv't prob meas.)

Fact (from Moore Ergodicity Thm)

Γ *is ergodic on* $\mathbf{F}$*:*

- Γ-*invariant meas'ble* ***function*** *is const a.e.*
- Γ-*invariant meas'ble* ***set*** *has measure* 0 *or* 1
- *(a.e. orbit is dense)*

Proof of key fact (amenability)

$\exists\ \psi \colon \mathbf{F} \to \operatorname{Prob}(S^1)$ Γ-equivariant, measurable.

$(\mathbf{F} = \{\, (\ell, \Pi) \mid \ell \subset \Pi \subset \mathbb{R}^3 \,\})$

$G = \mathrm{SL}(3,\mathbb{R})$ is *transitive* on $\mathbf{F}$.

So $\mathbf{F} \cong G/P$, where $P = \operatorname{Stab}_G(\text{flag}) = \begin{bmatrix} * & * & * \\ & * & * \\ & & * \end{bmatrix}$.

Want $\Psi \colon G \to \operatorname{Prob}(S^1)$, Γ-equi, s.t. $\Psi(gp) = \Psi(g)$.

Let $\mathcal{E} = \{\, \Gamma\text{-equivariant } \Psi \colon G \to \operatorname{Prob}(S^1) \,\}$.

- G acts on $\mathcal{E}$ by translation.
- $\operatorname{Prob}(S^1) \subset C(S^1)^*$ is compact, convex

  $\Rightarrow \mathcal{E}$ is a compact, convex set.

We want P to have a fixed point in $\mathcal{E}$.

We want P to have a fixed point in cpct, cnvx set $\mathcal{E}$.

Exercise

Group H is amenable:

- *H acts continuously on cpct metric space X*
  $\Rightarrow \exists$ H-invariant prob meas on X.
- *H acts linearly on cpct convex set $\mathcal{E}$* $\subset$ *Banach*
  $\Rightarrow \exists$ fixed point in $\mathcal{E}$.

Exercise

P is amenable.

Hint: Show P is solvable. (Recall: solvable groups are amenable.)

$$P = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} \triangleright \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \triangleright \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \triangleright \{\mathrm{Id}\}.$$

Abelian quotients

Key fact: $\exists\ \psi : \mathbf{F} \to \mathrm{Prob}(S^1)$ Γ-equi, measurable.

$(\mathbf{F} = \{\, (\ell, \Pi) \mid \ell \subset \Pi \subset \mathbb{R}^3 \,\})$

Ghys: ψ is constant a.e. $(\exists)$

Observation

Let $\psi_{\text{atom}}(x) =$ atomic part of $\psi(x)$, so

$$\psi(x) = \psi_{\text{atom}}(x) + \psi_{\text{no atom}}(x).$$

Then $\psi_{\text{atom}}, \psi_{\text{no atom}}$ are Γ-equi, measurable.

Only two cases:

1. $\psi(x)$ purely atomic, $\forall x$.
2. $\psi(x)$ has no atoms, $\forall x$.

No need to consider mixed case.

Case 1. $\psi(x)$ is purely atomic.

Assume $\psi\colon \mathbf{F} \to S^1$, so $\psi_3\colon \mathbf{F}^3 \to (S^1)^3$.

$$\psi_3(x,y,z) = \big(\psi(x),\psi(y),\psi(z)\big)$$

Circular order: $(S^1)^3 = X^+ \sqcup X^- \sqcup \{\text{singular}\}$.
X^+ is invariant under $\mathrm{Homeo}_+(S^1)$, so
$\psi_3^{-1}(X^+)$ is Γ-invariant subset of $\mathbf{F}^3$.
Contradiction if $\nexists$ Γ-invariant subsets.
More precisely, if Γ is *ergodic* on $\mathbf{F}^3$.

Not ergodic — Ghys works with *fibered product*
$\{\, (\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3) \in \mathbf{F}^3 \mid \Pi_1 = \Pi_2 = \Pi_3 \,\}$
instead of the cartesian product.

Recall. $\mathbf{F} \cong G/P$, where $G = \mathrm{SL}(3,\mathbb{R})$, $P = \begin{bmatrix} * & * & * \\ & * & * \\ & & * \end{bmatrix}$.

Theorem (Moore Ergodicity Thm)

Γ *is ergodic on* $G/H \iff H$ *is not compact.*

Corollary

Γ *is ergodic on* $\mathbf{F}$.

$$\mathrm{Stab}(\mathcal{F}) = P = \begin{bmatrix} * & * & * \\ & * & * \\ & & * \end{bmatrix}. \quad \text{Not compact.}$$

Corollary

Γ *is ergodic on* $\mathbf{F}^2 = \mathbf{F} \times \mathbf{F}$.

$$\mathrm{Stab}(\mathcal{F}_1, \mathcal{F}_2) = \begin{bmatrix} * & & \\ & * & \\ & & * \end{bmatrix}. \quad \text{Not compact.}$$

$$\mathrm{Stab}(\mathcal{F}_1, \mathcal{F}_2) = \begin{bmatrix} * & & \\ & * & \\ & & * \end{bmatrix}.$$

Theorem (Moore Ergodicity Thm)

Γ *is ergodic on* $G/H \iff H$ *is not compact.*

Corollary

Γ *is* **not** *ergodic on* $\mathbf{F}^3$*.*

$\mathrm{Stab}(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) = \{\pm \mathrm{Id}\}$ finite.

Corollary

Γ *is ergodic on* $\{(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) \in \mathbf{F}^3 \mid \Pi_1 = \Pi_2 = \Pi_3\}$*.*

$$\mathrm{Stab}_G(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) = \begin{bmatrix} \lambda & 0 & * \\ 0 & \lambda & * \\ 0 & 0 & 1/\lambda^2 \end{bmatrix} \text{ not compact.}$$

Case 2. $\psi(x)$ has no atoms.

- $\mathrm{Prob}_0(S^1) = \{\, \mu \in \mathrm{Prob}(S^1) \mid \mu$ has no atoms $\}$.
- $\psi_2 \colon \mathbf{F}^2 \to (\mathrm{Prob}_0(S^1))^2$ $(\psi_2(x,y) = (\psi(x),\psi(y)))$

  measurable, Γ-equivariant.
- $d \colon (\mathrm{Prob}_0(S^1))^2 \to \mathbb{R}$

  $$d(\mu_1,\mu_2) = \sup_{J \text{ interval}} |\mu_1(J) - \mu_2(J)|.$$

  continuous, Γ-invariant.

Composition $d \circ \psi_2$ is Γ-invariant; hence const a.e.:

$d(\psi_2(x,y)) = c$, for a.e. $x, y \in \mathbf{F}$.

Exercise

$c = 0$. *Hint:* $d(\psi_2(x,x)) = 0$. d *continuous, Lusin's Thm.*

Kazhdan's property (T)

Theorem

Thompson's group T does not have (T). [Reznikov]
In fact, T has Haagerup property. [Farley]

Recall: $T \subset \mathrm{Homeo}_+^{\mathrm{PL}}(S^1)$, $T \hookrightarrow \mathrm{Diff}_+^{\infty}(S^1)$ [Ghys Sergiescu]

Theorem (Navas)

$\Gamma \subset \mathrm{Diff}^2(S^1)$, *Kazhdan's property T $\Rightarrow$ Γ finite.*

Theorem (Navas)

$\Gamma \subset \mathrm{Diff}^2(S^1)$, *Kazhdan's property T* $\Rightarrow \Gamma$ *finite.*

Definition

Γ has *Kazhdan's property T*:

$$H^1(\Gamma, \mathcal{H}) = 0, \ \forall \text{ unitary } \Gamma\text{-module } \mathcal{H}.$$

I.e.: $\mathcal{H} = L^2(S^1 \times S^1)$ (square-integrable funcs).

$\mathcal{H}$ is Hilbert space (normed ∞-dim'l vec space)

$$\|F\|^2 = \left| \int_{S^1 \times S^1} F(s,t)^2 \, ds \, dt \right|$$

Spse Γ acts on $\mathcal{H}$ by unitaries $(\|F^g\| = \|F\|)$.

$\alpha \colon \Gamma \to \mathcal{H}$ is a 1-cocycle $(\alpha(gh) = \alpha(g)^h + \alpha(h))$

$\Rightarrow \alpha$ is a coboundary $(\exists v \in \mathcal{H}, \alpha(g) = v^g - v)$.

Navas: $\Gamma \subset \mathrm{Diff}^2(S^1)$, property T $\Rightarrow$ Γ finite.

For $F \in L^2(S^1 \times S^1)$ and $g \in \Gamma$,

$$F^g(r,s) = F\big(g(r), g(s)\big)\, |g'(r)|^{1/2}\, |g'(s)|^{1/2}.$$

This is unitary rep'n of Γ on $L^2(S^1 \times S^1)$.

Let $F(r,s) = f(r-s)$ on $S^1 \times S^1$, $f(x) = \frac{1}{x} + C^\infty$.

- $F \notin L^2(S^1 \times S^1)$ (bcs $1/x$ singularity)
- $\forall g \in \mathrm{Diff}^2(S^1)$, $F^g - F$ is bounded.

Define $\alpha(g) = F^g - F \in L^2(S^1 \times S^1)$.

α is a cocycle, so, by Kazhdan's Property T,

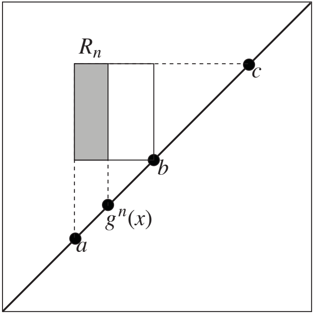
$$\exists\, v \in L^2(S^1 \times S^1),\ F^g - F = v^g - v.$$

Then $F - v$ is Γ-invariant and

$F - v \notin L^2(S^1 \times S^1)$ ($1/x$ singular on diag)

Let $\mu = (F - v)^2 \, dr \, ds$, so

- μ is Γ-invariant measure on $S^1 \times S^1$;
- $\mu(\text{rect}) = \begin{cases} \infty & \text{if touches diagonal} \\ \text{finite} & \text{if away from diagonal} \end{cases}$



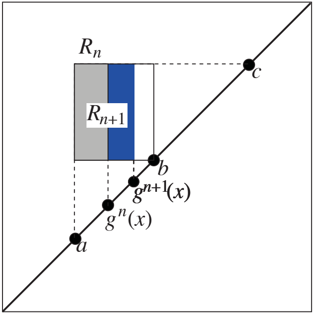
Choose $g \in \Gamma$, has a fixed pt.
Pass to triple cover,
so g has ≥ 3 fixed points:

$$g(a) = a, \; g(b) = b, \; g(c) = c.$$

For $x \in (a, b)$,

$$\lim g^n(x) = \begin{cases} b & \text{as } n \to \infty \\ a & \text{as } n \to -\infty \end{cases}$$

$$R_n = (a, g^n(x)) \times (b, c)$$



$$R_n = (a, g^n(x)) \times (b, c).$$
$$g(R_n) = (a, g^{n+1}(x)) \times (b, c)$$
$$= R_{n+1}.$$

$$g(R_n) = R_{n+1}$$
$$\Rightarrow \mu(R_n) = \mu(R_{n+1})$$
$$\Rightarrow \mu(R_{n+1} - R_n) = 0$$
$$\Rightarrow \mu\left(\bigcup_{n=-\infty}^{\infty} (R_{n+1} - R_n)\right) = 0$$
$$\Rightarrow \mu((a, b) \times (b, c)) = 0$$

$\rightarrow\leftarrow$ $(a, b) \times (b, c)$ touches the diagonal.

References

The Ghys Proof

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