Using left-invariant orders to study actions on 1-manifolds

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Lecture 2:
Amenability and the Ghys-Burger-Monod Theorem
Recall (assume \(G\) countable)

\[ G \subset \text{Homeo}_+([0,1]) \iff G \text{ has a left-inv't order.} \]

Theorem

\(G\) has a left-inv't order, amenable, f.g. \(\implies G \twoheadrightarrow \mathbb{Z}\).

Outline of Proof (3 easy steps, plus 1 fact).

1. \([\text{Ghys, Stepin}]\) \(G\) acts on space \(\mathcal{O}\) of left-inv't orders.
2. Amenability: \(\exists\) \(G\)-inv’t probability meas on \(\mathcal{O}\).
3. Poincaré Recurrence: there is a recurrent order.
4. Known: \(G\) has recurrent order \(\implies G \twoheadrightarrow \mathbb{Z}\). \(\checkmark\)
Step 1: idea of Ghys and Stepin

We know: $G$ has a left-invariant order.

Proposition

$\mathcal{O} := \{ \text{left-invariant orders on } G \} \neq \emptyset$.

- $\mathcal{O} \subset 2^{G \times G}$ is compact (and Hausdorff)

  Basic open set: $\mathcal{O}_{a_1,\ldots,a_r} = \{ \prec \mid a_1 \prec a_2 \prec \cdots \prec a_r \}$

- $G$ acts on $\mathcal{O}$ by right translation:

  \[ a \prec_g b \iff ag^{-1} \prec bg^{-1} \]

  This is an action by homeomorphisms.

Proof.

$\mathcal{O}_{a_1,\ldots,a_r} g = \mathcal{O}_{a_1g,\ldots,a_rg}$. 
We know: $G$ acts continuously on $\emptyset$.

**Definition**

$\mu$ *probability measure* on metric space $X$: $\mu(X) = 1$.

**Definition**

$G$ *amenable*:

$G$ acts (continuously) on cpct metric space $X$ 

$\Rightarrow \exists \ G$-invariant prob meas on $X$.

I.e., $\mu(g(A)) = \mu(A)$, $\forall g \in G$, $\forall A \subset X$.

$G$ *amenable* 

$\Rightarrow \exists \ G$-invariant prob meas on $\emptyset$. 
$G \text{ amenable } \iff \exists \text{ inv’t prob meas on any cpct } G\text{-space}$

**Exercises (using this definition)**

- **finite groups** are amenable  \[ \mu = \frac{1}{\#G} \sum_{g \in G} \delta_{g}x \]
- $\mathbb{Z}$ is amenable  \[ \mu = \lim_{k \to \infty} \frac{1}{N_k} \sum_{n=0}^{N_k} \delta_n(x) \]
- $(N \lhd G \text{ with } N, G/N \text{ amenable } \Rightarrow G \text{ amenable})$
- *(subgroups), quotients of amen grps are amen*  
- $(G \text{ amen } \iff \text{ every f.g. subgrp of } G \text{ is amen})$
- **solvable groups** are amenable
- (grps of subexponential growth are amenable)
- **free groups** are not amenable
**Step 3: Poincaré Recurrence Theorem**

We know: $\exists$ $G$-invariant probability measure on $\mathcal{O}$.

**Recall:** $G$ finite $\implies g^n = e$, $\exists \ n \to \infty$.

**Poincaré Recurrence Theorem**

$T$ measure-preserving homeomorphism of $\mathcal{O}$

$\implies$ a.e. $x$ is recurrent:

$\forall$ open $U \ni x$, $\exists n \to \infty$, $T^n(x) \in U$.

**Sketch of proof.**

Given $U$, show a.e. $x \in U$ returns to $U$ at least once.

Let $B = \{ b \in U \mid T^n(b) \notin U, \forall n > 0 \}$.

Spse $\mu(B) \neq 0$. $\mu(T^n(B))$ const $\implies$ not all disj.

$\exists x \in T^n(B) \cap B \subset T^n(B) \cap U = \emptyset$. 

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**Poincaré Recurrence Theorem**

A measure-preserving homeomorphism of $\Theta$ implies a.e. $x$ is **recurrent**:

$$\forall \text{ open } U \ni x, \exists n \to \infty, T^n(x) \in U.$$ 

**Corollary**

$$\forall g \in G, \text{ a.e. } \prec \text{ is recurrent for } g:$$

$$a_1 \prec a_2 \prec \cdots \prec a_r$$

$$\Rightarrow a_1 g^n \prec a_2 g^n \prec \cdots \prec a_r g^n, \quad \exists n \to \infty$$

**Exercise**

a.e. $\prec$ is recurrent for every element of $G$.

**Hint:** $G$ is ctble (f.g.). Ctble union of sets of meas 0.
$G$ has left-inv’t order, amenable, f.g. $\implies G \to \mathbb{Z}$.

**Conjecture (Linnell)**

*Can replace amenable with no free subgroups.*

**Question (Ghys)**

¿ *Can replace amenable with Haagerup property?*

**Definition**

Say $G$ is a *Poincaré-Recurrence group* if:

$\forall$ (continuous) action of $G$ on compact, metric $X$, 
$\exists x \in X$, $x$ is recurrent for all cyclic subgroups of $G$.

1. ¿ $G$ no free subgroups $\implies G$ is P-R group?
2. ¿ $\exists$ torsion-free P-R group that is not amenable?
Theorem (Ghys-Burger-Monod)

Let $\Gamma = \text{finite-index subgrp of } \text{SL}(3, \mathbb{Z})$ (or any lattice in $\text{SL}(3, \mathbb{R})$). If $\Gamma \hookrightarrow \text{Homeo}_+(S^1)$, then $\exists$ finite orbit.

Conjecture

Every orbit is finite.

Apply Reeb-Thurston Stability Theorem:

Corollary (Ghys-Burger-Monod)

Let $\Gamma = \text{finite-index subgrp of } \text{SL}(3, \mathbb{Z})$ (or any lattice in $\text{SL}(3, \mathbb{R})$). Then $\Gamma \not\hookrightarrow \text{Diff}^1(S^1)$. 
Assume
\[ \Gamma \to \text{Homeo}_+ (S^1), \quad \Gamma = \text{SL}(3, \mathbb{Z}) \quad \text{(for simplicity)} \]

Key fact (amenability)
\[ F = \text{Furstenberg boundary} = \text{flag variety} = \{ (\ell, \Pi) | \ell \subset \Pi \subset \mathbb{R}^3 \} \]
\[ \Rightarrow \exists \Gamma\text{-equivariant, random map } F \to S^1. \]
\[ \psi : F \to \text{Prob}(S^1) \quad \Gamma\text{-equivariant, meas’ble.} \]

Proof of Ghys
Show \( \psi \) constant. (Then \( \psi(\Gamma) \) is \( \Gamma \)-invariant probability measure.)
Proof of key fact (amenable)

\[ \exists \psi : F \to \text{Prob}(S^1) \text{ } \Gamma\text{-equivariant, measurable.} \]
\[ (F = \{ (\ell, \Pi) | \ell \subset \Pi \subset \mathbb{R}^2 \}) \]

\[ G = \text{SL}(3, \mathbb{R}) \text{ is } \text{transitive} \text{ on } F. \]
\[ \text{So } F \cong G/P, \text{ where } P = \text{Stab}_G(\text{flag}) = \begin{bmatrix}
* & * & * \\
* & * & * \\
* & & *
\end{bmatrix}. \]

Want \( \Psi : G \to \text{Prob}(S^1), \Gamma\text{-equi, s.t. } \Psi(gp) = \Psi(g). \)
Let \( \mathcal{E} = \{ \Gamma\text{-equivariant } \Psi : G \to \text{Prob}(S^1) \}. \)

- \( \text{Prob}(S^1) \subset C(S^1)^* \) is compact, convex
  \[ \Rightarrow \mathcal{E} \text{ is a compact, convex set.} \]
- \( G \) acts on \( \mathcal{E} \) by translation.

We want \( P \) to have a fixed point in \( \mathcal{E} \).
We want $P$ to have a fixed point in cpct, cvnx set $\mathcal{E}$.

**Exercise**

**Group $H$ is amenable:**

- $H$ acts continuously on cpct metric space $X$ \[ \Rightarrow \exists H\text{-invariant prob meas on } X. \]
- $H$ acts linearly on cpct convex set $\mathcal{E} \subset \text{Banach}$ \[ \Rightarrow \exists \text{ fixed point in } \mathcal{E}. \]

**Exercise**

$P$ is amenable.

*Hint:* Show $P$ is solvable.  (Recall: solvable groups are amenable.)

\[
P = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} > \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} > \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} > \{\text{Id}\}. \] Abelian quotients
Amenable groups with a left-invariant order


Ghys’ proof


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<th>The Burger-Monod Proof</th>
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