Locally symmetric subspaces of locally symmetric spaces

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The results for semisimple Lie groups

Proposition (well known)

G simple Lie group $\implies$ $G \dot\supset \mathrm{SO}(3)$ *or* $\mathrm{SL}(2,\mathbb{R})$.
$\mathbb{R}$-rank $G \geq 2$ $\implies$ $G \dot\supset \mathrm{SO}(2,3)$ *or* $\mathrm{SL}(3,\mathbb{R})$.
G semisimple, $\mathbb{R}$-rank $G \geq 2$
$\implies$ $G \dot\supset \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ *or* $\mathrm{SL}(3,\mathbb{R})$.

Category of semisimple Lie grps with $\mathbb{R}$-rank ≥ 2 has only 2 minimal elements (up to local isomorphism).

Geometric version:

Category of symmetric spaces G/K with rank ≥ 2 has only 2 minimal elements (up to isometry).

The result for lattices

Recall

Subgroup Γ of G is a *lattice*:
Γ is *discrete*, and G/Γ has *finite volume.*
Irreducible: $\Gamma \dot{\neq} \Gamma_1 \times \Gamma_2$ (both factors infinite)

E.g., $\mathrm{SL}(n,\mathbb{Z})$ is an (irred) lattice in $\mathrm{SL}(n,\mathbb{R})$.

Definition

Assume Γ is an irreducible lattice in G, where
- G is semisimple, with $\mathbb{R}$-rank $G \geq 2$, and
- G/Γ is *not* compact.

Then Γ is a (non-cocompact) *higher-rank lattice*.

Remark

The category of higher-rank latts has *no* min'l el'ts:
every subgrp of finite index is a higher-rk latt.

Definition

Higher-rank lattice is *almost-minimal* if
no subgroup of ∞ index is a higher-rank lattice.

Theorem (Chernousov-Lifschitz-Morris)

Γ *almost-minimal of higher rank* $\implies$
Γ *is a lattice in* $\mathrm{SL}(3,\mathbb{R})$, *or* $\mathrm{SL}(3,\mathbb{C})$,
or a product of $\mathrm{SL}(2,\mathbb{R})$*'s and* $\mathrm{SL}(2,\mathbb{C})$*'s.*

E.g., $\mathrm{SL}(2,\mathbb{Z}[\sqrt{2}])$ is irred latt in $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$.

Theorem (Chernousov-Lifschitz-Morris)

Γ *almost-minimal of higher rank* $\implies$ Γ *is a lattice in*
$\mathrm{SL}(3,\mathbb{R})$, $\mathrm{SL}(3,\mathbb{C})$, *or* $\mathrm{SL}(2,\mathbb{R})^m \times \mathrm{SL}(2,\mathbb{C})^n$.

Note: Every higher-rank lattice contains an almost-minimal one.

Corollary (by Mostow-Margulis Rigidity)

Γ = *higher rank lattice in* G
$\implies$ $\exists\, H \subset G$, $H \cap \Gamma$ *is an irreducible lattice in* H,
$H \in \{\mathrm{SL}(3,\mathbb{R}),\ \mathrm{SL}(3,\mathbb{C}),\ \mathrm{SL}(2,\mathbb{R})^m \times \mathrm{SL}(2,\mathbb{C})^n\}$.

Key point

The possibilities for H are *classical*, not exceptional.

Geometric version

Locally symmetric space of higher rank

- X = locally symm space (complete, noncpct type, finite volume) $= \Gamma\backslash G/K$,
- rank $X \geq 2$,
- X is irreducible (not $X_1 \times X_2$, up to finite cover),
- X is **not** compact.

X contains a locally symm subspace of higher rank (closed, totally geodesic) whose universal cover is:
- a product of $\mathbb{H}^2$s and $\mathbb{H}^3$s (with ≥ 2 factors), or
- $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ or $\mathrm{SL}(3,\mathbb{C})/\mathrm{SU}(3)$.

Theorem (Chernousov-Lifschitz-Morris)

Γ *irred, noncocpct latt in* G, $\mathbb{R}$-rank $G \geq 2$ $\implies$ $\Gamma \dot\supset$
latt in $\mathrm{SL}(3,\mathbb{R})$ *or* $\mathrm{SL}(3,\mathbb{C})$ *or* $\mathrm{SL}(2,\mathbb{R})^m \times \mathrm{SL}(2,\mathbb{C})^n$.

Key to Proof (Margulis Arithmeticity Thm)

Γ is *arithmetic subgrp* of G: $\Gamma \doteq G \cap \mathrm{SL}(\ell,\mathbb{Z}) = G_{\mathbb{Z}}$.
($\exists\, G \dot\hookrightarrow \mathrm{SL}(\ell,\mathbb{R})$ with $G_{\mathbb{Q}}$ dense in G. $G_{\mathbb{Q}}$ is a *$\mathbb{Q}$-form* of G.)

Corollary

It suffices to show $G_{\mathbb{Q}}$ *contains either:*
- $\mathrm{SL}(2,F)$, *($\exists$ extension F of $\mathbb{Q}$,* not imag quad*), or*
- *some* ("isotropic") *$\mathbb{Q}$-form of* $\mathrm{SL}(3,\mathbb{R})$, *or*
- *some* ("isotropic") *$\mathbb{Q}$-form of* $\mathrm{SL}(3,\mathbb{C})$.

We wish to show $G_{\mathbb{Q}}$ contains $\mathrm{SL}(2,F)$ or ...

Theorem (classical)

There is a "list" of all the $\mathbb{Q}$-forms $G_{\mathbb{Q}}$:
- $\mathrm{SL}(n,F)$, $\mathrm{SO}(B;F)$, $\mathrm{Sp}(n,F)$, $\mathrm{SU}(B,\tau;F)$,
- $\mathrm{SL}(n,D)$, *where D is a division algebra over F,*
- $\mathrm{SU}(B,\tau;D)$,
- *a few other "exceptional" or "triality" cases.*

Consider each possible $\mathbb{Q}$-form case-by-case.

Example

$\mathrm{SL}(n,D) \supset \mathrm{SL}(n, \text{max'l subfield } F) \supset \mathrm{SL}(2,F)$.

Work in progress

[CLM]: the almost-minimal elements in the category of *non-cocompact* higher-rank lattices.
Lattices in $\mathrm{SL}(3,\mathbb{R})$, $\mathrm{SL}(3,\mathbb{C})$, $\mathrm{SL}(2,\mathbb{R})^m \times \mathrm{SL}(2,\mathbb{C})^n$

Now considering the *cocompact* case
with Skip Garibaldi and Ben McReynolds.
- Answer not as nice:
  - lattices in many groups of type A_n:
    $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{SL}(n,\mathbb{C})$, $\mathrm{SL}(n,\mathbb{H})$, $\mathrm{SU}(m,n)$.
  - *Maybe* also some latts in $\mathrm{SO}(2,6)$ (in progress)
- No others are almost-minimal
  (no exceptional or triality)