Introduction to Ratner's Theorems on Unipotent Flows

Dave Witte Morris

University of Lethbridge, Alberta, Canada
http://people.uleth.ca/~dave.morris
Dave.Morris@uleth.ca

UBC, March 31, 2010

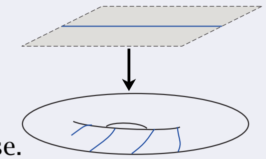
Abstract

Let f be the obvious covering map from Euclidean n-space to the n-torus. It is well known that if L is any straight line in n-space, then the closure of $f(L)$ is a very nice submanifold of the n-torus. In 1990, Marina Ratner proved a beautiful generalization of this observation that replaces Euclidean space with any Lie group G, and allows L to be any subgroup of G that is "unipotent." We will discuss the statement of Ratner's Theorem, and a few of its important consequences. Topological and geometric aspects will be emphasized, while algebraic technicalities will be pushed to the background.

Elementary example

Let M = torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$

- covering map $f\colon \mathbb{R}^2 \to M$
- L = line in $\mathbb{R}^2$.



If the slope of L is irrational, it is classical that $\overline{f(L)}$ is dense.

Exercise. Let M = n-torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$

- covering map $f\colon \mathbb{R}^n \to M$
- L = vector subspace of $\mathbb{R}^n$.

Closure $\overline{f(L)} = f(S)$ is a torus $\mathbb{T}^k$ ($\exists$ subspace S of $\mathbb{R}^n$).

The closure of $f(L)$ is a very nice submanifold of M.

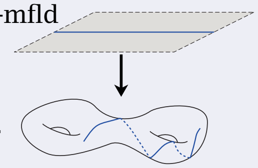
Example in geometry

Example

Let M = compact, hyperbolic 3-mfld

- covering map $f\colon \mathbb{H}^3 \to M$
- line $\mathbb{H}^1 \hookrightarrow \mathbb{H}^3$.

Closure $\overline{f(\mathbb{H}^1)}$ can be a fractal.



Consequence of Ratner's Thm

$\mathbb{H}^2 \subset \mathbb{H}^3 \implies \overline{f(\mathbb{H}^2)} = f(\mathbb{H}^k)$ is a submfld of M

(immersed, maybe not embedded).

(Similar for other locally symmetric spaces.)

Recall

- $f\colon \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$
- L = vector subspace of $\mathbb{R}^n$

$\implies \overline{f(L)} = f(S)$, $\exists$ vector subspace S of $\mathbb{R}^n$.

- $\mathbb{R}^n$ is a Lie group.
  I.e., a group (under vector addition) and a manifold, and the group operations are smooth.
- The subgroup $\mathbb{Z}^n$ is closed and discrete.

Definition

Let G = any Lie group, Γ = closed, discrete subgroup.
Then G/Γ is a manifold, a *homogeneous space.*
It is best if G/Γ is compact.
(Or allow G/Γ to have finite volume: Γ is a *lattice*.)

Recall

- $f\colon \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$
- L = vector subspace of $\mathbb{R}^n$

$\implies \overline{f(L)} = f(S)$, $\exists$ vector subspace S of $\mathbb{R}^n$.

Generalization (Ratner's Theorem)

Replace:

- $\mathbb{R}^n$ with any Lie group G
- $\mathbb{Z}^n$ with any lattice Γ in G
- L with any subgroup of G
  that is generated by "unipotent" elements
- S with a closed subgroup of G

Applications

Example ([N. Shah])

$M = \mathbb{H}^3/\Gamma$, $f\colon \mathbb{H}^3 \to M$, $\mathbb{H}^2 \subset \mathbb{H}^3$
$\implies \overline{f(\mathbb{H}^2)} = f(\mathbb{H}^k)$ is a submfld of M

Idea of proof.

$\mathbb{H}^2 \subset \mathbb{H}^3$ corresponds to $\mathrm{SO}(1,2) \subset \mathrm{SO}(1,3)$.

- $\mathrm{SO}(1,2)^\circ$ is gen'd by unip el'ts, and
- $\mathrm{SO}(1,2)$ is a maximal subgroup. □

Example. Let Γ_1 and Γ_2 be conjugates of $\mathrm{SL}(n,\mathbb{Z})$.
Then $\overline{\Gamma_1 \cdot \Gamma_2} = \mathrm{SL}(n,\mathbb{R})$ unless $\Gamma_1 \doteq \Gamma_2$.

Γ_1 and Γ_2 can be any lattices in any simple Lie group G

Corollary ("Oppenheim Conj." [Margulis])

Let Q be a real quadratic form in $n \ge 3$ variables
(e.g., $x^2 - \sqrt{2}xy + \sqrt{3}z^2$).
Then $Q(\mathbb{Z}^n)$ is dense in $\mathbb{R}$
unless $\approx \mathbb{Z}$-coefficients, or definite, or degenerate.

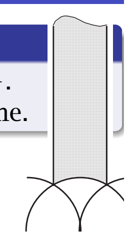
Proof for $n = 3$.

Let $G = \mathrm{SL}(3,\mathbb{R})$, $\Gamma = \mathrm{SL}(3,\mathbb{Z})$, and
$L = \mathrm{SO}(Q) = \{\, \ell \in \mathrm{SL}(3,\mathbb{R}) \mid Q(\ell\vec{x}) = Q(\vec{x})\,\}$.
Ratner: $\overline{L\Gamma} = S\Gamma$, some subgroup $S \subset G$.
Fact: L is maximal in G, so $S = L$ or G.
$S = L \implies Q \approx \mathbb{Z}$-coefficients
So $L\Gamma$ is dense in G. Therefore
$\overline{Q(\mathbb{Z}^3)} \supset Q(\overline{L\Gamma}\mathbb{Z}^3) = Q(G\mathbb{Z}^3) = Q(\mathbb{R}^3) = \mathbb{R}$. □

Idea of proof of Ratner's Theorem

Example

$G = \mathrm{SL}(2,\mathbb{R}) = \{\ 2 \times 2 \text{ real mat's of det } 1\ \}$.
Let $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. Then G/Γ has finite volume.



Other choices of Γ can make G/Γ compact.

Definition

Define $u^t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ and $a^t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$.

Each is a homomorphism from $\mathbb{R}$ to $\mathrm{SL}(2,\mathbb{R})$.
u^t is a **unipotent** one-parameter subgroup.

Polynomial divergence

$$u^t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad a^t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$$

$d(x, qx) = ||q||$. $\qquad d(u^t x, u^t q x) = ||u^t q u^{-t}||$.

$$u^t q u^{-t} = \begin{bmatrix} \alpha + \gamma t & \beta + (\delta - \alpha)t - \gamma t^2 \\ \gamma & \delta - \gamma t \end{bmatrix}$$

Points move apart at polynomial speed. *(Slowly!)*

$$u^t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \quad u^t q u^{-t} = \begin{bmatrix} \alpha + \gamma t & \beta + (\delta - \alpha)t - \gamma t^2 \\ \gamma & \delta - \gamma t \end{bmatrix}$$

Points move apart at polynomial speed. *(Slowly!)*

Contrast: $a^t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \quad a^t q a^{-t} = \begin{bmatrix} \alpha & \beta e^{2t} \\ \gamma e^{-2t} & \delta \end{bmatrix}$

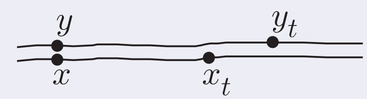
Points move apart at exponential speed. *(Quickly!)*

Shearing

$$u^t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \; u^t q u^{-t} = \begin{bmatrix} \alpha + \gamma t & \beta + (\delta - \alpha)t - \gamma t^2 \\ \gamma & \delta - \gamma t \end{bmatrix}$$

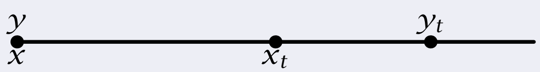
Shearing

Fastest motion is parallel to the orbits.



Shearing

Fastest motion is parallel to the orbits.

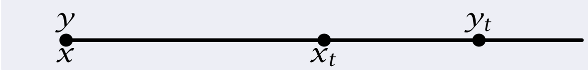


Contrast: $a^t q a^{-t} = \begin{bmatrix} \alpha & \beta e^{2t} \\ \gamma e^{-2t} & \delta \end{bmatrix}$

Fastest motion is transverse to the orbits.

Shearing

Fastest motion is parallel to the orbits.



Key idea in Ratner's proof

Ignore motion along the orbit, and look at the *transverse* motion perpendicular to the orbit.

Example

$$u^t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \; u^t q u^{-t} = \begin{bmatrix} \alpha + \gamma t & \beta + (\delta - \alpha)t - \gamma t^2 \\ \gamma & \delta - \gamma t \end{bmatrix}$$

Fastest motion is along $\{u^t\}$.

Ignoring this, largest terms are diagonal (in $\{a^t\}$)

Observation

$a^t \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} a^{-t} = \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}$: $\quad$ *a^t normalizes $\{u^t\}$.*

Proposition

For action of a unipotent subgroup, the fastest transverse divergence is along the normalizer.

Proposition

For action of a unipotent subgroup, the fastest transverse divergence is along the normalizer.

Application

If $C = \overline{\{u^t\}x}$ is a min'l closed, u^t-inv't subset of G/Γ, $\;$ and $\{ g \in G \mid gC = C \} = \{u^t\}$, then $C = \{u^t\}x$.

Proof.

Choose $x, y \in C$ with $x \approx y$.

Then $u^s x \approx g\, u^{s'} y$ with $g \in N_G(\{u^t\}) \setminus \{u^t\}$.

Pretend $u^s x = g\, u^{s'} y \in C \cap gC$.

But $u^t gC = g u^{t'} C = gC$, so $C \cap gC$ is u^t-inv't.

Minimality: $gC = C$. Therefore $g \in \{u^t\}$. $\rightarrow\leftarrow$ □

Ratner's Theorems

Ratner's Theorem on Orbit Closures

$\overline{\{u^t\}x} = Sx$, $\quad \exists$ subgroup S of G.

Ratner's Equidistribution Theorem

$\{u^t\}x$ is *equidistributed* in Sx:

$$\frac{1}{T}\int_0^T f(u^t x)\, dt \to \int_{Sx} f\, d\mu \quad \text{for } f \in C_c(G/\Gamma)$$

where μ = (normalized) S-invariant volume form on Sx.

Ratner's Measure-Classification Theorem

Any ergodic u^t-invariant probability measure on G/Γ is (normalized) S-invariant volume form on Sx for some subgroup S and some $x \in G/\Gamma$.

Further reading

chapter of my forthcoming book on arithmetic grps

- free PDF file on my web page
  http://people.uleth.ca/~dave.morris /books/IntroArithGroups.html

my book: *Ratner's Theorems on Unipotent Flows*

- free PDF file on my web page (or the arxiv)
  http://people.uleth.ca/~dave.morris /books/Ratner.html
- US$20 from University of Chicago Press