On hamiltonian paths in 2-generated solvable Cayley digraphs

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Assumptions

- *G is a finite group*
- *S is a generating set for G*
- *p is prime*

Definition

$\overrightarrow{\mathrm{Cay}}(G;S)$ = directed graph

- vertices: elements of G
- directed edge $v \to vs$ for $v \in G$ and $s \in S$

$S = S^{-1} \implies$ undirected graph $\mathrm{Cay}(G;S)$

Conjecture (≈1970)

$\mathrm{Cay}(G;S)$ *has a hamiltonian cycle.*

Exercise (Rankin, 1948)

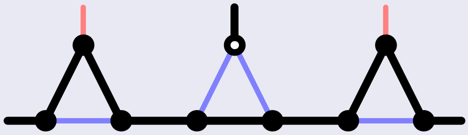
$\overrightarrow{\mathrm{Cay}}(\mathbb{Z}_k \oplus \mathbb{Z}_\ell; (1,0), (0,1))$ *has no hamiltonian cycle*
if $\gcd(k,\ell) = 1$ *(and many other cases)*

Exercise (Nathanson?, ≈1976?)

G abelian $\implies \overrightarrow{\mathrm{Cay}}(G;S)$ *has a hamiltonian path.*

Exercise (J. Milnor?, ≈1975?)

$\overrightarrow{\mathrm{Cay}}(G;a,b)$ *no ham path*
if $|a| = 2$, $|b| = 3$, *and* $|ab^2| < |G|/9$.



Exercise (J. Milnor?, ≈1975?)

$\overrightarrow{\mathrm{Cay}}(G; a, b)$ *no ham path*
if $|a| = 2$, $|b| = 3$, *and* $|ab^2| < |G|/9$.

Example (≈Milnor)

No ham path in $\overrightarrow{\mathrm{Cay}}(\mathbb{Z}_6 \ltimes \mathbb{Z}_p;\ (3,0),(2,1))$
if $p > 9$ and $p \equiv 1 \pmod 6$

Remark

These are Cayley digraphs on *solvable* groups,
so solvable Cayley digs need not have ham paths.

Example (≈Milnor)

No ham path in $\overrightarrow{\text{Cay}}(\mathbb{Z}_6 \ltimes \mathbb{Z}_p;\ (3,0),(2,1))$
if $p > 9$ and $p \equiv 1 \pmod 6$

Proposition (Morris, in preparation)

$\exists\ \overrightarrow{\text{Cay}}(\mathbb{Z}_k \ltimes \mathbb{Z}_p;\ a, b)$ *with no ham path*
if $3 < p \equiv 3 \pmod 4$.

Remark

- Assumption $3 < p$ is necessary [in preparation]
  $|[G,G]| = 2$ or $3 \implies \exists$ ham path in $\overrightarrow{\text{Cay}}(G;S)$.
- Assumption $p \equiv 3 \pmod 4$ not necessary?
  [in progress]
- $|a|$ and $|b|$ are arbitrarily large (unlike Milnor)

- $\nexists$ ham path in $\overrightarrow{\mathrm{Cay}}(\mathbb{Z}_k \ltimes \mathbb{Z}_p;\ a, b)$ if ...
- G solvable $\not\Rightarrow$ $\exists$ ham path in $\overrightarrow{\mathrm{Cay}}(G;S)$
- G abelian $\implies$ $\exists$ ham path in $\overrightarrow{\mathrm{Cay}}(G;S)$

abelian $\subset$ nilpotent $\subset$ solvable

Open question

¿ *G nilpotent* $\implies$ $\exists$ *ham path in* $\overrightarrow{\mathrm{Cay}}(G;S)$?

Theorem (Witte, 1986)

$|G| = p^n \implies \exists$ *ham path in* $\overrightarrow{\mathrm{Cay}}(G;S)$

Theorem (Morris, 2011)

G nilpotent $\implies$ $\exists$ *ham path in* $\overrightarrow{\mathrm{Cay}}(G;a,b)$

Theorem (Morris, 2011)

G nilpotent $\implies$ $\exists$ *ham path in* $\overrightarrow{\mathrm{Cay}}(G; a, b)$.

Lemma

$H \subset G \quad \implies \quad \exists\, H = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_n \triangleleft G$, *s.t.*

- H_i/H_{i-1} *is a* p*-group* $\quad \exists p = p_i$
- H_i/H_{i-1} *is generated by a conjugate of* H

Let $H = \langle a^{-1}b \rangle$. Assume $H_0 = H_n = H$. I.e., $H \triangleleft G$.
G/H gen'd by a, so (a^k) is ham cyc in $\overrightarrow{\mathrm{Cay}}(G/H; S)$.
Then $H = \langle a^k, a^k(a^{-1}b) \rangle = \langle a^{k-1}S \rangle$.
H abel $\implies$ $\overrightarrow{\mathrm{Cay}}(H; a^{k-1}S)$ has ham path $(a^{k-1}s_i)_{i=1}^{m}$.
Ham path in $\overrightarrow{\mathrm{Cay}}(G; S)$ is $(a, a, \ldots, a, s_i)_{i=1}^{m+1}$ $\sharp$. $\square$

D. W. Morris: 2-generated Cayley digraphs on nilpotent groups have hamiltonian paths
http://arxiv.org/abs/1103.5293

Š. Miklavič and P. Šparl: On Hamiltonicity of circulant digraphs of outdegree three, *Discrete Math.* 309 (2009), no. 17, 5437–5443.