Hamiltonian cycles in circulant graphs and digraphs

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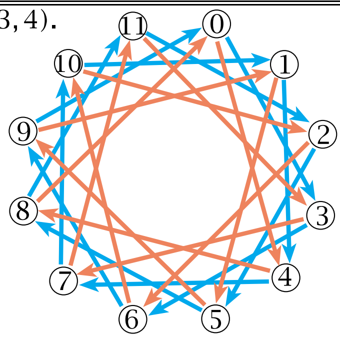
Abstract

We will discuss the status of the search for hamiltonian cycles in circulant graphs and circulant digraphs. Circulant graphs have many hamiltonian cycles, but recent joint work with Joy Morris and David Moulton uncovered a nontrivial parity condition that restricts the hamiltonian cycles in certain cases. A different parity condition arose in joint work with Stephen Locke that constructed infinitely many circulant digraphs with no hamiltonian cycles. The case of digraphs remains largely open.

Eg. $\mathrm{Circ}(n;S) = \mathrm{Circ}(12;3,4)$.

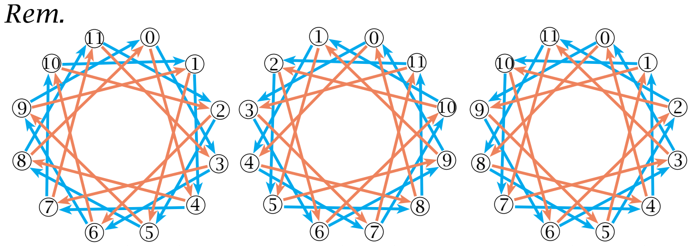


Defn. Circulant digraph $\mathrm{Circ}(n;S)$ $\quad n \in \mathbb{Z}^+$, $S \subset \mathbb{Z}$.

- vertices = elements of $\mathbb{Z}_n$
- directed edge from x to $x+s$ $\quad$ for $s \in S$.

(Notation: $x \xrightarrow{s} y$ if $y = x + s$.)

Rem.

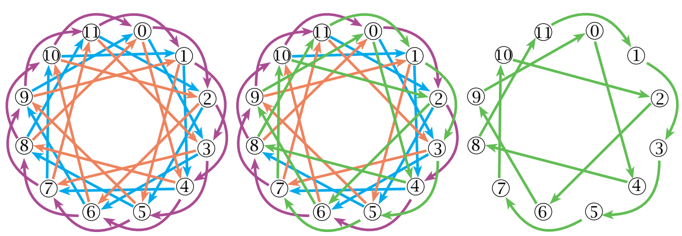


$\mathrm{Circ}(n;S) \quad = \quad \mathrm{Circ}(n;S) \quad \cong \quad \mathrm{Circ}(n;-S)$

Lem. $\mathrm{Circ}(n;S) \cong \mathrm{Circ}(n;rS)$ *if* $\gcd(r,n) = 1$.
(Notation: $\mathrm{Circ}(n;S) \underset{\times r}{\cong} \mathrm{Circ}(n;rS)$.)

Proof. $x \xrightarrow{s} y$ in $\mathrm{Circ}(n;S) \Leftrightarrow rx \xrightarrow{rs} ry$ in $\mathrm{Circ}(n;rS)$.

Eg. A *Hamiltonian cycle* in $\mathrm{Circ}(12;2,3,4)$.



$0 \xrightarrow{4} 4 \xrightarrow{4} 8 \xrightarrow{3} 11 \xrightarrow{2} 1 \xrightarrow{2} 3 \xrightarrow{2} 5 \xrightarrow{2} 7 \xrightarrow{3} 10 \xrightarrow{4} 2 \xrightarrow{4} 6 \xrightarrow{3} 9 \xrightarrow{3} 0$

Defn. Hamiltonian cycle:

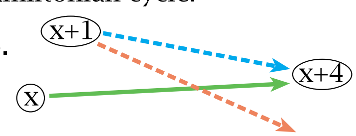
- visit each vertex exactly once
- end the same place you started.

(Must travel correct direction on each edge.)

Prop. $\nexists$ *hamiltonian cycle in* $\mathrm{Circ}(12;3,4)$.

Proof. Suppose H is a hamiltonian cycle.

Say vertex x travels by 4.



Then $x+1$ cannot travel by 3 (collision at $x+4$).
So $x+1$ must travel by 4.
By induction, every vertex travels by 4. $\rightarrow\leftarrow$

So no vertex travels by 4. They all travel by 3. $\rightarrow\leftarrow$

Also: $\nexists$ ham cyc in $\mathrm{Circ}(n;a,b)$ if $b-a$ rel prime to n and $\gcd(a,n) \neq 1$ and $\gcd(b,n) \neq 1$.

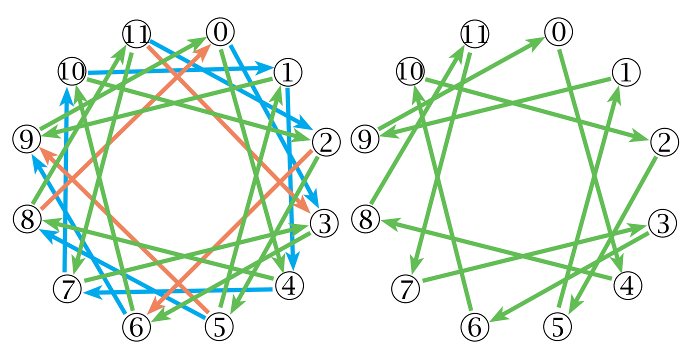
The undirected case.

Exer. Every circulant *graph* has a hamiltonian cycle.
(Ignore the directions on the edges.)

In fact, $\mathrm{Cay}(n;S)$ has *many* hamiltonian cycles (usually).

Assumption. $\mathrm{Circ}(n;S)$ is *connected*.
I.e., $\gcd(n, s_1, s_2, \ldots, s_k) = 1$.

Eg. $\mathrm{Circ}(12;3,4)$ has a ham cyc if we ignore directions.



$0 \xrightarrow{4} 4 \xrightarrow{4} 8 \xrightarrow{3} 11 \xrightarrow{-4} 7 \xrightarrow{-4} 3 \xrightarrow{3} 6 \xrightarrow{4} 10 \xrightarrow{4} 2 \xrightarrow{3} 5 \xrightarrow{-4} 1 \xrightarrow{-4} 9 \xrightarrow{3} 0$

Ques (Alspach).
Are there restrictions on the weights of ham cycs?

Eg. Consider $\mathrm{Circ}(12;3,4)$. Put
- weight 1 on each 3-edge,
- weight 2 on each 4-edge.

Let H be the hamiltonian cycle from above:
$0 \xrightarrow{4} 4 \xrightarrow{4} 8 \xrightarrow{3} 11 \xrightarrow{-4} 7 \xrightarrow{-4} 3 \xrightarrow{3} 6 \xrightarrow{4} 10 \xrightarrow{4} 2 \xrightarrow{3} 5 \xrightarrow{-4} 1 \xrightarrow{-4} 9 \xrightarrow{3} 0$

$\mathrm{wt}(H) = 2+2+1-2\ -2\ +1+2+2+1-2\ -2\ +1$
$= 4.$

Eg. Consider $\mathrm{Circ}(12;3,4)$.
Let weight of every edge be $\pm 1 \equiv 1 \pmod 2$.

For any cycle C, $\mathrm{wt}(C) \equiv$ #edges in $C \pmod 2$.
- C: $0 \xrightarrow{4} 4 \xrightarrow{4} 8 \xrightarrow{4} 0 \quad \Rightarrow \mathrm{wt}(C) \equiv 3 \not\equiv 0 \pmod 2$.
- H = any ham cyc $\Rightarrow H$ has 12 edges
  $\Rightarrow \mathrm{wt}(H) \equiv 12 \equiv 0 \pmod 2$.

This is a (nontrivial) restriction on the wts of ham cycs:
- some cycles have weight $\not\equiv 0$,
- but every ham cyc has weight $\equiv 0$.

Similar restriction for $\mathrm{Circ}(n;S)$ if n is even
and some cycle has an odd number of edges.
Let's call this the *bipartiteness restriction.*

Thm (Morris^2-Moulton, Alspach-Locke-Witte, L-W).
$\nexists$ nontrivial restrictions on the weights of ham cycs
(except perhaps the bipartiteness restriction)
if n is odd or $\#(S \cup -S) \notin \{3,4\}$.
I.e., for any weighting w and any integer m
$\exists$ cycle C, s.t. $\mathrm{wt}(C) \not\equiv 0 \pmod m$,
$\Rightarrow \exists$ ham cycle H, s.t. $\mathrm{wt}(H) \not\equiv 0 \pmod m$.

Rem. We found all restrictions (usually none).
I.e., we determined which flows are sums of ham cycles.
Usually, every (even) flow.

Eg. Consider an exceptional case with $\#(S \cup -S) = 4$.

Thm (Morris^2-Moulton). *Consider* $\mathrm{Circ}(n;a,b)$. *Assume*
- *a is odd,*
- *b is even,*
- *$n \equiv 2 \pmod 4$.*

Put weight
- *0 on each a-edge $x \xrightarrow{a} x+a$,*
- *$(-1)^x$ on edge $x \xrightarrow{b} x+b$.*

Then $\mathrm{wt}(\text{ham cyc}) \equiv 0 \pmod 4$.

(Cycle $\quad 0 \underset{1}{\xrightarrow{b}} b \underset{0}{\xrightarrow{a}} a+b \underset{1}{\xrightarrow{-b}} a \underset{0}{\xrightarrow{-a}} 0 \quad$ has weight $2 \not\equiv 0$.)

There are *many* hamiltonian cycles;
not obvious that $\mathrm{wt}(H)$ is always divisible by 4.

Use a more geometric definition of wt(even cycle).

Define "*imbalance*" of C:
- vertices *not* on C colored black/white (certain way),
- $\mathrm{imb}(C) = (\#\text{black}) - (\#\text{white}) \pmod 4$.

Key fact. $\mathrm{wt}(C) \equiv \mathrm{len}(C) + \mathrm{imb}(C) - 2 \pmod 4$.
- Prove for directed cycles.
- Any cycle can be modified to a directed cycle
  without changing $\mathrm{wt}(C) - (\mathrm{len}(C) + \mathrm{imb}(C) - 2)$.

Proof of Thm. $\mathrm{wt}(\text{ham cyc}) \equiv 0 \pmod 4$.
- $\mathrm{len}(H) = n \equiv 2 \pmod 4$.
- $\mathrm{imb}(H) = 0 \quad$ (#black = 0 = #white).

$\Rightarrow \mathrm{wt}(H) \equiv 2\ +0\ -2 = 0.$ □

The directed case.

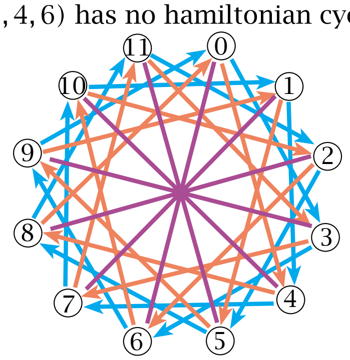
Recall.
- $\mathrm{Circ}(12;3,4)$ has no ham cyc.
- $\mathrm{Circ}(n;a,b)$ has no ham cyc if $\gcd(a-b,n) = 1$.

Thm (Rankin). $\mathrm{Circ}(n;a,b)$ *has a ham cyc* $\Leftrightarrow$
$\exists\, s,t \in \mathbb{Z}^{\geq 0}$, *s.t.* $s+t = \gcd(a-b,n) = \gcd(sa+tb,n)$.

Conj. $\mathrm{Circ}(n;S)$ *has a hamiltonian cycle if* $\#S > 3$.

Problem. *Does* $\mathrm{Circ}(n;a,b,c)$ *have a hamiltonian cycle?*

Eg. $\mathrm{Circ}(12;3,4,6)$ has no hamiltonian cycle.



Eg. $\mathrm{Circ}(n;a,b,c)$ with no ham cyc and $n < 48$.

Circ(12; 2, 3, 8)	Circ(30; 2, 6, 21)	Circ(40; 4, 5, 24)
Circ(12; 3, 4, 6)	Circ(30; 2, 9, 24)	Circ(42; 2, 3, 24)
Circ(18; 2, 3, 12)	Circ(30; 2, 10, 25)	Circ(42; 2, 6, 27)
Circ(18; 2, 6, 15)	Circ(30; 3, 10, 18)	Circ(42; 2, 7, 28)
Circ(20; 2, 5, 12)	Circ(30; 5, 6, 20)	Circ(42; 2, 12, 33)
Circ(24; 2, 3, 14)	Circ(36; 2, 3, 20)	Circ(42; 2, 15, 36)
Circ(24; 2, 9, 12)	Circ(36; 2, 9, 20)	Circ(42; 2, 18, 39)
Circ(24; 3, 4, 16)	Circ(36; 2, 15, 20)	Circ(42; 3, 14, 24)
Circ(28; 2, 7, 16)	Circ(36; 3, 8, 18)	Circ(42; 6, 7, 28)
Circ(30; 2, 3, 18)	Circ(40; 2, 5, 22)	Circ(44; 2, 11, 24)

n is even. Sometimes $n/2 \in S$:

Circ(12; 3, 4, 6) Circ(24; 2, 9, 12) Circ(36; 3, 8, 18)

Circ(12; 3, 4, 6)   Circ(24; 2, 9, 12)   Circ(36; 3, 8, 18)

$n = 12k$

$\mathrm{Circ}(12;3,4,6) \underset{\times(-1)}{\cong} \mathrm{Circ}(12;9,8,6)$
$= \mathrm{Circ}(12;6,8,9)$

$\mathrm{Circ}(24;2,9,12) \underset{\times 7}{\cong} \mathrm{Circ}(24;14,15,12)$
$= \mathrm{Circ}(24;12,14,15)$

$\mathrm{Circ}(36;3,8,18) \underset{\times 7}{\cong} \mathrm{Circ}(36;21,20,18)$
$= \mathrm{Circ}(36;18,20,21)$

Thm (Locke-Witte).
$\mathrm{Circ}(12k;\, 6k,\, 6k+2,\, 6k+3)$ *has no hamiltonian cycle.*

Circ(12; 2, 3, 8)	Circ(18; 2, 6, 15)	~~Circ(24; 2, 9, 12)~~
~~Circ(12; 3, 4, 6)~~	Circ(20; 2, 5, 12)	Circ(24; 3, 4, 16)
Circ(18; 2, 3, 12)	Circ(24; 2, 3, 14)	Circ(28; 2, 7, 16)

$\mathrm{Circ}(12;2,3,8) \qquad 12/2 = 6 = 8-2.$

$\mathrm{Circ}(18;2,3,12) \qquad 18/2 = 9 = 12-3.$
$\mathrm{Circ}(18;2,3,12) \underset{\times -1}{\cong} \mathrm{Circ}(18;16,15,6)$
$= \mathrm{Circ}(18;15,16,6) \qquad 6-15 = -9 \equiv 9.$

$\mathrm{Circ}(18;2,6,15) = \mathrm{Circ}(18;6,2,15) \qquad 2-6 \equiv 14.$
$2-15 \equiv 5. \qquad 5\times 11 = 55 \equiv 1 \pmod{18}$
$\mathrm{Circ}(18;2,6,15) \underset{\times 11}{\cong} \mathrm{Circ}(18;4,12,3) = \mathrm{Circ}(18;3,4,12)$

$\mathrm{Circ}(2k;\, a, a+1, a+k)$

$\mathrm{Circ}(2k;\, a, a+1, a+k)$

Circ(12; 2, 3, 8)	Circ(18; 3, 4, 12)	~~Circ(24; 2, 9, 12)~~
~~Circ(12; 3, 4, 6)~~	Circ(20; 14, 15, 4)	Circ(24; 20, 21, 8)
Circ(18; 15, 16, 6)	Circ(24; 2, 3, 14)	Circ(28; 6, 7, 20)

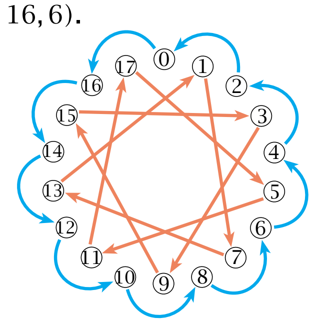
Prop. $\mathrm{Circ}(2k; a, a+1, a+k)$ *has no ham cycle* $\Rightarrow$
- $\gcd(a, 2k) \neq 1$,
- $\gcd(a+1, 2k) \neq 1$,
- *$a+k$ is even.*

Thm (Locke-Witte). *Converse is true.*

Eg. No ham cyc in Circ(18; 15, 16, 6).

Eg. No ham cyc in Circ(18; 15, 16, 6).

Define $G \subset \mathrm{Circ}(18;15,16,6)$:
- x even $\Rightarrow x \xrightarrow{16} x+16$ in G,
- x odd $\Rightarrow x \xrightarrow{6} x+6$ in G.



Note. G consists of cycles:
- $0 \xrightarrow{16} 16 \xrightarrow{16} 14 \xrightarrow{16} 12 \xrightarrow{16} 10 \xrightarrow{16} 8 \xrightarrow{16} 6 \xrightarrow{16} 4 \xrightarrow{16} 2 \xrightarrow{16} 0$
- $1 \xrightarrow{6} 7 \xrightarrow{6} 13 \xrightarrow{6} 1$
- $3 \xrightarrow{6} 10 \xrightarrow{6} 16 \xrightarrow{6} 3$
- $5 \xrightarrow{6} 11 \xrightarrow{6} 17 \xrightarrow{6} 5$

The number of cycles is *even.*

Define $G \subset \mathrm{Circ}(18;15,16,6)$:
- x even $\Rightarrow x \xrightarrow{16} x+16$ in G,
- x odd $\Rightarrow x \xrightarrow{6} x+6$ in G.

Note. x travels by 16 $\Leftrightarrow$ $x+9$ travels by 6.

Key fact. H any ham cyc $\Rightarrow$
x travels by 16 $\Leftrightarrow$ $x+9$ travels by 6 or 15.

Lem. *Any* $G_1, G_2 \subset \mathrm{Circ}(18;15,16,6)$
s.t. x *travels by* 16 $\Leftrightarrow$ $x+9$ *travels by* 6 *or* 15
(*and each* x *has one edge in and one edge out*)
$\Rightarrow$ *#cycles in* $G_1 \equiv$ *#cycles in* $G_2 \pmod 2$.

#cycles in $G \equiv 0$.   #cycles in $H = 1$. $\rightarrow\leftarrow$

Research Problems

Conj. $\mathrm{Circ}(n;S)$ *has a hamiltonian cycle if* $\#S > 3$.

Problem. $\exists\, \mathrm{Circ}(n;a,b,c)$ *with no ham cycle and n odd?*

Problem. *Locke-Witte: infinitely many examples of* $\mathrm{Circ}(n;a,b,c)$ *with no ham cycle. Are there others?*

(Only verified completeness to $n = 100$.)

Problem. *Does* $\mathrm{Circ}(n;a,b,-b)$ *always have a ham cyc if* $b \neq n/2$?

References

B. Alspach, The search for long paths and cycles in vertex-transitive graphs and digraphs, in: *Combinatorial Mathematics VIII* (ed. K.L. McAvaney), Lecture Notes in Mathematics, Vol. 884, Springer-Verlag, Berlin (1981) 14–22.

B. Alspach, S. C. Locke, and D. Witte: The Hamilton spaces of Cayley graphs on abelian groups, *Discrete Math.* 82 (1990) 113–126.

C. C. Chen and N. F. Quimpo: On strongly hamiltonian abelian group graphs, in: K.L. McAvaney, ed., *Combinatorial Mathematics VIII*, Lecture Notes in Mathematics, Vol. 884 (Springer-Verlag, Berlin, 1981) 23–34.

S.J. Curran and J.A. Gallian, Hamiltonian cycles and paths in Cayley graphs and digraphs — a survey, *Discrete Math.*, 156 (1996) 1–18.

S. J. Curran and D. Witte: Hamilton paths in cartesian products of directed cycles, *Ann. Discrete Math.* 27 (1985) 35–74.

S. C. Locke and D. Witte: Flows in circulant graphs of odd order are sums of Hamilton cycles, *Discrete Math.* 78 (1989) 105–114.

D. Morris, J. Morris and D. P. Moulton: Flows that are sums of hamiltonian cycles in abelian Cayley graphs (to appear in *Discrete Math.*) `http://arxiv.org/abs/math.CO/0309050`

R. A. Rankin: A campanological problem in group theory. *Proc. Cambridge Philos. Soc.* 44, (1948). 17–25.

R. A. Rankin: A campanological problem in group theory. II. *Proc. Cambridge Philos. Soc.* 62 1966 11–18.

D. Witte and J.A. Gallian, A survey: hamiltonian cycles in Cayley digraphs, *Discrete Math.*, 51 (1984) 293–304.