Hamiltonian paths and cycles
in vertex-transitive graphs and digraphs

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Abstract
It was conjectured, more than 30 years ago, that every vertex-transitive graph has a hamiltonian path, and that every Cayley graph has a hamiltonian cycle (unless the graph is disconnected). This talk will survey the progress that has been made on these problems, both of which remain very much open. For example, it is still not known whether every Cayley graph on every dihedral group has a hamiltonian cycle. (The cubic case was settled by Brian Alspach and C.-Q. Zhang.) Related questions on directed graphs will also be discussed; for example, it is easy to see that every (connected) circulant digraph has a hamiltonian path, but we do not know which circulant digraphs have hamiltonian cycles.

Eg. Cartesian product \( C_m \square C_n \).

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<tr>
<td>( (x, y) ) ( \rightarrow (x \pm 1, y) )</td>
<td>( (x, y) ) ( \rightarrow (x, y \pm 1) )</td>
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Edges \( v \rightarrow v \pm e_1 \) and \( v \rightarrow v \pm e_2 \),
where \( e_1 = (1,0) \) and \( e_2 = (0,1) \)
are the natural generators of \( \mathbb{Z}_m \oplus \mathbb{Z}_n \).

In fact, Cay(\( G ; S \)) has many Hamilton cycles.

Thm (Chen-Quimpo). Cay(\( G ; S \)) is Hamilton connected
(i.e., \( \forall v, w, \exists \) Ham path from \( v \) to \( w \))
unless valence \( \leq 2 \) (or graph is bipartite).

Conj (Alspach). Cay(\( G ; S \)) is Hamilton decomposable
(i.e., edge-disjoint union of Ham cycles [+ 1-factor?]).

Thm (Bermond-Favaron-Mahéo). True if valence \( \leq 5 \).

In fact, Cay(\( G ; S \)) has many Hamilton cycles.

Recall. Any flow in any graph is a sum of cycles.

- A flow is a function \( \phi : E^\pm(X) \rightarrow \mathbb{Z} \), s.t. . . .
- Flows can be added. \( (\phi + \psi)(e) = \phi(e) + \psi(e) \)
- Any directed cycle defines a flow \( \phi : E^\pm(X) \rightarrow \{0, \pm 1\} \).

Thm (Alspach-Locke-Witte, Locke-Witte).
Every flow in Cay(\( G ; S \)) is a sum of Hamilton cycles
if \#G is odd (and Cay(\( G ; S \)) \( \not\cong C_4 \square C_3 \)).

In fact, Cay(\( G ; S \)) has many Hamilton cycles.

\#G even \( \Rightarrow \) Ham cycs are even flows: \( \sum_{e \in E^+(X)} f(e) \in 2\mathbb{Z} \).
#\(G\) even \(\Rightarrow\) Ham cycs are even flows: \[ \sum_{e \in E^+ (X)} f(e) \in 2\mathbb{Z}. \]

Converse:

**Thm** (Morris, Morris, Moulton).
*Every even flow in Cay\((G;S)\) is a sum of Hamilton cycles if valence \(\ge 5\).*

We have *almost* finished classifying the counterexamples, but there is still an open case of valence 4.

**Conj** (Moulton).
*Not every even flow is a sum of Ham cycs in \(C_{odd} \square C_{4k+2}\).*

**Conj.** Cay\((G;S)\) has a Hamilton cycle.

**Problem.** Show Cay\((G;S)\) has a Ham cyc if \(G\) is small (say, \(\#G < 100\)).

Frank Ruskey: *cubic of order < 100 may be done? (Gordon Royle’s web page has a list.)*

**Problem.** Find Ham cyc in prism Cay\((G;S)\) \(\square P_2\).

*Done for cubic case:*

**Thm** (Durnberger, Marušić, Keating-Witte).
*Cay\((G;S)\) has a Hamilton cycle if \([G,G]\) has prime order or, more generally, is cyclic of prime-power order.*

**Problem.** Find Hamilton cycle if \([G,G] \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2\).

**Thm** (Durnberger, Marušić, Keating-Witte).
*Cay\((G;S)\) has a Hamilton cycle if \([G,G]\) has prime order.*

**Idea of proof.** \(\mathcal{G} = G/[G,G]\) is abelian \(\Rightarrow\) Cay\((\mathcal{G};S)\) has a Ham cyc \(\mathcal{C}\).

Lift \(\mathcal{C}\) to a path \(P\) in Cay\((G;S)\).

**Assume** \(P\) is not a cycle. (**“Marušić’s method”**)

Then we construct Ham cyc in Cay\((G;S)\) by concatenating translates of \(P\). \(\square\)

**Summary.** Cay\((G;S)\) has a Hamilton cycle if
- \([G,G]\) is cyclic of prime-power order, or
- \(G\) is of prime-power order.

Most groups do not fall into these categories.

**Eg.** Dihedral group \(D_{2n}\) of order \(2n\)
- \(\langle t, f \mid t^n = e, f^2 = e, ftf = t^{-1}\rangle\)
- symmetries of a regular \(n\)-gon
- \(n\) rotations \((t^0, t^1, ..., t^{n-1})\)
- \(n\) reflections \((f, ft^1, ft^2, ..., ft^{n-1})\)

**Rem.** Cay\((D_{2n};\{t,f\}) \cong \) prism (over cycle \(\langle t \rangle\)) has a Hamilton cycle.

**Thm** (Witte). Cay\((G;S)\) has a Hamilton cycle if \#\(G\) is a prime power \(p^n\).

**Problem.** Find Hamilton cycle if \#\(G\) = 2\(p^n\).

**Problem.** Find Hamilton cycle if \(G = P \times Q\) where \#\(P\) and \#\(Q\) are prime powers.

\((G\) is “nilpotent.”)

**Problem.** Generalize to vertex-transitive graphs.

**Thm** (Yu Qing Chen).
*Vertex-transitive graphs of order \(p^4\) have Ham cycs.*

**Eg.** Dihedral group \(D_{2n}\) of order \(2n\)
- \(\langle t, f \mid t^n = e, f^2 = e, ftf = t^{-1}\rangle\)

**Rem.** Cay\((D_{2n};\{t,f\})\) has a Hamilton cycle.

**Ezer.** If gcd\((a,b,n) = 1\), then \(\langle f, ft^a, ft^b \rangle = D_{2n}\), so Cay\((D_{2n};\{f,ft^a,ft^b\})\) is cubic.

(Embeds on torus, with every face a hexagon.)

**Thm** (Alspach-Zhang).
*Cay\((D_{2n};\{f,ft^a,ft^b\})\) has a Hamilton cycle.*

**Conj.** Cay\((D_{2n};\{f,ft^{a_1},ft^{a_2},...,ft^{a_r}\})\) has a Ham cyc.
(Then Cay\((D_{2n}, S)\) has a Ham cyc, for any \(S\).)

**Conj.** Cay\((D_{2n},\{f,ft^{a_1},ft^{a_2},...,ft^{a_r}\})\) has Ham cyc.

Possible approach (stronger induction hypothesis):

**Problem** (Alspach).
*Show Cay\((D_{2n}, S)\) is Hamilton connected if \#\(S\) = 3."

Another approach is via Cayley digraphs. **(directed)**

**Defn.** \(G\) = finite group
\(S\) = generating set of \(G\)

**Cayley digraph** Cay\((G;S)\):
vertices = elements of \(G\)
edge \(v \to vs\) for \(v \in G\) and \(s \in S\).
**Exer.** If $G$ is abelian, then $\overrightarrow{\text{Cay}}(G; S)$ has a Hamilton path.

**Exer.** $\overrightarrow{\text{Cay}}(\mathbb{Z}_{12}; \{3, 4\})$ does not have a Hamilton cycle.

**Problem.** Which circulant digraphs have a Ham cycle?

**Answer** (Rankin, 1948) when $\#S = 2$ (and $G$ is abelian).

**Conj.** (Curran-Witte). If
- $G$ is cyclic (or abelian), and
- $\#S \geq 3$, and
- $S$ is minimal (no proper subset of $S$ generates $G$),
then $\overrightarrow{\text{Cay}}(G; S)$ has a Hamilton cycle.

**Exer.** Conj $\Rightarrow \overrightarrow{\text{Cay}}(D_{2n}, S)$ has a Ham cycle, for every $S$.

**Conj.** (Curran-Witte). $G$ abelian, $\#S \geq 3$, $S$ minimal $\Rightarrow \overrightarrow{\text{Cay}}(G; S)$ has a Hamilton cycle.

**Conj.** If $\overrightarrow{C}_1, \ldots, \overrightarrow{C}_r$ are directed cycles, and $r \geq 3$, and $\text{gcd}(n_1, \ldots, n_r) = 1$, then $\overrightarrow{C}_1 \Box \cdots \Box \overrightarrow{C}_r$ is Ham conn.
  (I.e., $\forall$ vertices $v, w, \exists$ Ham path from $v$ to $w$.)

**Exer.** In $\overrightarrow{\text{Cay}}(\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_r}; S_{\text{nat}})$, if there is a Hamilton path from $v$ to $w$, then

$$w_1 + \cdots + w_r \equiv v_1 + \cdots + v_r - 1 \pmod{\text{gcd}(n_1, \ldots, n_r)}.$$

**Conj.** (Austin-Gavlas-Witte). Converse if $r \geq 3$.

**Thm.** (Austin-Gavlas-Witte). True when $n_1 = \cdots = n_r$.

**Survey articles.**

B. Alspach,
The search for long paths and cycles in vertex-transitive graphs and digraphs,
*Combinatorial mathematics, VIII* (Geelong, 1980), pp. 14–22,
MR 83b:05080

D. Witte and J.A. Gallian,
A survey: Hamiltonian cycles in Cayley graphs,
MR 86a:05084

S.J. Curran and J.A. Gallian,
Hamiltonian cycles and paths in Cayley graphs and digraphs—a survey,
MR 97f:05083

**Thm.** (Alspach). $\overrightarrow{\text{Cay}}(G; \{x, y\})$ has a Hamilton cycle if $\langle x \rangle$ is a normal subgroup of $G$.

**Thm.** (Rankin). $\overrightarrow{\text{Cay}}(G; \{x, y\})$ has a Hamilton cycle if $xy^{-1}$ has order 2 (i.e., $(xy^{-1})^2 = e$).

**Cor.** $G$ simple $\Rightarrow \exists x, y \in G$, $\overrightarrow{\text{Cay}}(G; \{x, y\})$ has $H$ cyc.

**Cor.** (Pak). $\forall G, \exists S, \overrightarrow{\text{Cay}}(G; S)$ has a Hamilton cycle, and $\#S \leq \log_2 \#G$.

**Conj.** $\forall G$, $\exists S$, $\overrightarrow{\text{Cay}}(G; S)$ has a Hamilton cycle, and $S$ is a minimal generating set (or minimum?).

There are many results on specific generating sets.