Abstract. It was conjectured about 40 years ago that every connected Cayley graph has a Hamiltonian cycle. This is easy to prove for Cayley graphs on abelian groups, but we are nowhere near a proof of the general case. The talk will discuss some of the progress that has been made, and some of the many open problems.

Notation
- $G = \text{finite group}$
- $S = \text{generating set for } G$
- $\text{Cay}(G; S) = \text{Cayley graph}$
  - vertices = elements of $G$
  - edge $g \rightarrow gs$ for $g \in G$ and $s \in S$

Conjecture (~1970)
$\text{Cay}(G; S)$ has a Hamiltonian cycle.

Not much progress.

Example
$\text{Cay}(\mathbb{Z}_m \times \mathbb{Z}_n; \{(1, 0), (0, 1)\})$ has a Hamiltonian cycle.

Proof.

Exercise: $\text{Cay}(G; S)$ has a Hamiltonian cycle if $G$ is abelian.

Example
$G = \text{dihedral group } D_{2n} = \langle f, t \mid f^2 = t^n = e, ftf = t^{-1} \rangle$
$S = \{f, ft^a, ft^b\}$
$\text{Cay}(D_{2n}; f, ft^a, ft^b)$ is hexagonal tiling on a torus.

This has a Hamiltonian cycle [Alspach-Qiang]. (But not easy!)

Conjecture
- $\text{Cay}(G; S)$ has a Hamiltonian cycle.
- $\text{Cay}(G; S)$ has a Hamiltonian path.
- $\text{Cay}(G; S)$ has a path of length $e \neq G$.
- $\text{Cay}(G; S)$ has a Hamiltonian cycle for some (irredundant) $S$.
- [Babai] Opposite conjecture: not always a ham path.

Proposition
- [Babai] $\exists \text{path (cycle) of length } \approx \sqrt{|G|}$.
- [Pak] $\forall G, \exists S, \text{Cay}(G; S)$ has a ham cyc, and $\#S \leq \log_2 \#G$.
- [Witte] $\forall S, \exists S', \text{Cay}(G; S')$ has a ham cyc, and $\#S' \leq \left(\#S\right)^2$.

Theorem
$\text{Cay}(G; S)$ has a Hamiltonian cycle if:
- $G$ is dihedral and $4 \mid \#G$. [Alspach et al.]
- $\#G = p^n$ (prime power). [Witte]
- commutator subgroup of $G$ is cyclic of prime-power order. [Keating-Witte]

Problem
Find a Hamiltonian cycle if:
- $G$ is dihedral.
- $G = P \times Q$ where $\#P$ and $\#Q$ are prime powers. ($G$ is “nilpotent.”)
The directed case

Conjecture. Cay(G; S) has a Hamiltonian cycle.

Remark
Cayley digraph Cay(G; S) might not have a Hamiltonian cycle or path.

Example. \( \exists \) Hamiltonian cycle in \( \overrightarrow{\text{Cay}}(\mathbb{Z}_{12}; 3, 4) \).

Proof.
Suppose \( H \) is a Hamiltonian cycle. Say vertex \( x \) travels by 4.
Then \( x + 1 \) cannot travel by 3 (collision at \( x + 4 \)).
So \( x + 1 \) must travel by 4.
By induction, every vertex travels by 4. \( \rightarrow \)
So no vertex travels by 4. They all travel by 3. \( \rightarrow \)

Theorem (Rankin, \( \sim 1940 \))
\[
\text{Cay}(\mathbb{Z}_n; a, b) \text{ has a } \text{ham cyc} \iff \\
\exists s, t \in \mathbb{Z}^{\geq 0}, \text{ s.t. } s + t = \gcd(a - b, n) = \gcd(sa + tb, n).
\]

Similar if we replace \( \mathbb{Z}_n \) with any abelian group.

Conjecture
\( G \) is abelian, \#S \( \geq 3 \), but \( \overrightarrow{\text{Cay}}(G; S) \) has no Hamiltonian cycle \( \Rightarrow \)
- \#S = 3,
- \( G \) is cyclic,
- \#G is even,
- \( S \) has an explicit description.

Example (Locke-Witte)
\( \overrightarrow{\text{Cay}}(\mathbb{Z}_{12k}; 6k, 6k + 2, 6k + 3) \) has no Hamiltonian cycle.

References
- D. Witte and J. A. Gallian,
  A survey: Hamiltonian cycles in Cayley graphs,
  MR 86a:05084
- S. J. Curran and J. A. Gallian,
  Hamiltonian cycles and paths in Cayley graphs and digraphs—a survey,
  MR 97f:05083
- B. Alspach,
  The search for long paths and cycles
  in vertex-transitive graphs and digraphs,
  *Combinatorial mathematics, VIII* (Geelong, 1980),
  Lecture Notes in Math. #884,
  MR 83b:05080
- I. Pak and R. Radoičić,
  Hamiltonian paths in Cayley graphs,
  MR2548568
- S. C. Locke and D. Witte,
  On non-Hamiltonian circulant digraphs of outdegree three,
  *J. Graph Theory* 30 (1999), no. 4, 319–331.
  MR1669452 (99m:05069)
- Brian Alspach, C. C. Chen, and Matthew Dean,
  Hamilton paths in Cayley graphs on generalized dihedral groups
  *Ars Mathematica Contemporanea* 3 (2010)