Groups of polynomial growth (after Gromov, Kleiner, and Shalom-Tao)

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Abstract. M. Gromov proved in 1981 that every group of polynomial growth has a nilpotent subgroup of finite index. This is a fundamental result in Geometric Group Theory, but Gromov’s proof is too difficult to include in many standard courses on the subject. A tremendous simplification of the proof of the key lemma was achieved by B. Kleiner in 2010 (using methods of T. H. Colding and W. P. Minicozzi II). A further improvement by Y. Shalom and T. Tao has made the theorem truly accessible. We present the simplified proof.

Geometric Group Theory

Γ = group (e.g., \( \mathbb{Z}^2 \))
S = symmetric finite generating set
(e.g., \( \{ (\pm 1, 0), (0, \pm 1) \} \))

\[ \forall x, y \in \Gamma: \quad y = x \cdot \frac{r^d}{2^d} = x s_1 s_2 \cdots s_k, \quad \exists s_i \in S, \ \exists k \in \mathbb{N} \]

Metric: \( d(x, y) \) = least such \( k \).

So \( \Gamma \) has both algebraic structure (group) and geometric structure (metric space).

Basic question: How are they related?

Gromov’s Thm

Definition
- ball \( B(r) = \{ x \in \Gamma \mid d(x, e) \leq r \} \)
- \( \Gamma \) has polynomial growth if \# \( B(r) \) \( \leq r^d \) (3d \( \in \mathbb{N} \)) (does not depend on choice of generating set)

Example
\( \Gamma = \mathbb{Z}^2: \#B(r) = r^2. \) So \( \mathbb{Z}^2 \) has polynomial growth.
- Easy: abelian grps have polynomial growth.
- Exer: Nilpotent grps have polynomial growth.
- Exer: Spse \( H \) is finite-index subgroup of \( \Gamma \).
  \( \Gamma \) has poly growth \( \iff H \) has poly growth.
- Cor: Virtually nilpotent grps have poly growth.
  Virtually _______ \( \iff \) finite index subgroup is ________.

Cor. Virtually nilp groups have polynomial growth.

Example
\( \Gamma = \text{free group } F_2. \)
\( S = \{ a^{\pm 1}, b^{\pm 1} \}. \)
\# \( B(r) \) is exp1 \( \gg r^d. \)
So \( F_2 \) does not have polynomial growth.

Exer: Groups of polynomial growth are amenable.

Theorem (Gromov)

Groups of polynomial growth are virtually nilpotent.

Assume: \( \Gamma \) has polynomial growth.
Prove: \( \Gamma \) is nilpotent \(* solvable*\! (polycyclic)

Exer. polycyclic with poly growth \( \Rightarrow \) nilpotent*.

Key Lem. \( \exists \) f.d. rep \( \rho: \Gamma \to \text{GL}(V), \) s.t. \#\( \rho(\Gamma) = \infty. \)


Idea: Let \( V = \{ f: \Gamma \to \mathbb{R} \}. \quad (f^\theta)(x) = f(\theta x) \)
This representation is not finite-dimensional. Kleiner: \( \exists \) finite-dim’l invariant subspace.
Let \( V_0 = \{ \text{Lipschitz, harmonic } f: \Gamma \to \mathbb{R} \}. \)
- \( |f(x) - f(y)| \leq C d(x, y), \quad \exists C \in \mathbb{R} \)
- \( f(x) = \sum_{s \in S} f(xs) \)

Theorem (Kleiner, Colding-Minicozzi)

\( \dim V_0 < \infty \) if \( \Gamma \) has polynomial growth.

Prop. \( \exists \) \( f \in V_0 \) not constant. (do not need poly growth)
So \( f^\Gamma \) is infinite. (nonconstant harmonic func has no max)

Thm. Cor. \( \dim \{ \text{Lipschitz, harmonic } f \} < \infty. \)

Defn. For \( \chi \subseteq \Gamma, \) \( \| f \|_{2, B(r)} \leq \text{poly} \cdot r\).
Polynomials have bounded doubling:
\( \varphi(r) \sim r^d \implies \varphi(2r) \leq C \varphi(r), \ \forall r. \)

Technical issue that we will ignore:
\( \varphi(r) \leq r^d \implies \text{bdd doubling for } \text{most } r, \text{not all } r. \)

Theorem \( \exists \epsilon, k, \forall r, \quad f \text{ harmonic } \implies \| f \|_{2, B(6kr)} \geq \epsilon k \| f \|_{2, B(kr)} \)
if \( f \) has mean 0 on each ball in a certain set \( B \)
balls of radius \( r \) with \#\( B = C(k) \).

This contradicts bounded doubling if \( \dim \gg \).

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Groups of polynomial growth are virtually nilpotent.
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Prop. \( \exists \) \( f \in V_0 \) not constant. (do not need poly growth)
So \( f^\Gamma \) is infinite. (nonconstant harmonic func has no max)
Let $f : \Gamma \to \mathbb{R}$.
- $\delta_s f(x) = f(xs) - f(x)$. 
- $|\delta_s f| \leq C$: $f$ is Lipschitz.
- $\delta f(x) = #S \cdot f(x) - \sum_{s \in S} f(xs)$.
- $\Delta f = 0$: $f$ is harmonic.

(Assume $S$ is symmetric (closed under inverses): $S^{-1} = S$.)

### Thm. 1: Poincaré Inequality

1. $\|f\|_{L^2(B(r))} \leq C_1 r^2 \sum_{s \in S} \|\delta_s f\|_{L^2(B(3r))}$

### Step 2 (Reverse Poincaré Inequality)

$$\sum_{s \in S} \|\delta_s f\|^2_{L^2(B(3r))} \leq C_2 \|f\|^2_{L^2(B(3r))}/R^2$$

### Idea of proof.

1. Poincaré ≤: $\|f\|^2_{L^2(B(r))} \leq C_1 r^2 \sum_{s \in S} \|\delta_s f\|^2_{L^2(B(3r))}$

### Step 3 (finish): $\|f\|^2_{L^2(B(6r))} \geq C_1 r^2 \sum_{s \in S} \|\delta_s f\|^2_{L^2(B(3r))}$

### Vitali Covering Lemma:

- Cover $B(kr)$ with $\frac{\#B(2kr)}{\#B(r/2)} < C'$ balls of radius $r$.
- No pt in more than $\frac{\#B(2kr)}{\#B(r/2)} < C$ tripled balls.

### Step 1 (Poincaré Inequality)

$$\|f\|^2_{L^2(B(r))} \leq C_1 r^2 \sum_{s \in S} \|\delta_s f\|^2_{L^2(B(3r))}$$

### Review of Kleiner’s proof

- Assume $\Gamma$ has polynomial growth.
- Poincaré ≤: $\|f\|^2_{L^2(B(r))} \leq C' r^2 \sum_{s \in S} \|\delta_s f\|^2_{L^2(B(3r))}$
- Reverse P I: $\sum_{s \in S} \|\delta_s f\|^2_{L^2(B(3r))} \leq C' \frac{r^2}{R^2} \|f\|^2_{L^2(B(3r))}$
- Thm. $\|f\|^2_{L^2(B(6r))} \geq C_1 r^2 \sum_{s \in S} \|\delta_s f\|^2_{L^2(B(3r))}$

### Cor. dim $\{\text{Lipschitz, harmonic } f \} < \infty$.

### Prop. $\exists$ Lipschitz, harmonic, nonconstant $f$.

### Key Lem.

- $\exists$ f.d. rep $\rho : \Gamma \to \text{GL}(V)$, s.t. $\#\rho(\Gamma) = \infty$.
- Gromov’s Theorem. $\Gamma$ is virtually nilpotent.

### Step 1 (Poincaré Inequality)

$$\|f\|^2_{L^2(B(r))} = \frac{1}{\#B(r)} \sum_{x \in B(r)} \|f(x)\|^2$$

- $\sum_{x \in B(r)} \|f(x) - \frac{1}{\#B(r)} \sum_{y \in B(r)} f(y)\|^2$
- $\leq \frac{1}{\#B(r)} \sum_{x \in B(r)} \sum_{y \in B(r)} |f(x) - f(y)|^2$
- $\leq \frac{1}{\#B(r)} \sum_{g \in B(2r)} \sum_{y \in B(r)} |g| y - f(y)|^2$
Step 1 (Poincaré Inequality)

\[ \| f \|_{2,B(R)}^2 \leq C_1 r^2 \sum_{s \in S} \| \partial_s f \|_{2,B(3r)}^2 \quad \text{if mean 0 on } B(r) \]

\[ \| f \|_{2,B(R)}^2 \leq \frac{1}{\#B(R)} \sum_{y \in B(R)} \sum_{y \in B(R)} |f(yg) - f(y)|^2. \]

\[ e = g_0 g_1 g_2 \cdots s_n g_n = g \]

\[ \sum_{y \in B(R)} |f(yg) - f(y)|^2 \leq 2r \cdot 2r \sum_{s \in S} \| \partial_s f \|_{2,B(3r)}^2 \]

\[ \| f \|_{2,B(R)}^2 \leq \frac{1}{\#B(R)} \cdot #B(2r) \cdot (2r)^2 \sum_{s \in S} \| \partial_s f \|_{2,B(3r)}^2 \]

Reverse P I: \[ \sum_{s \in S} \| \partial_s f \|_{2,B(R)}^2 \leq \frac{C}{R^2} \| f \|_{2,B(3r)}^2 \]

\[ \langle f' \rangle \langle (f^2)^2 \rangle = \| f' \|_{2,R}^2 + \frac{C}{R^2} \langle f' \rangle_{B(2R)} + \frac{C}{R} \langle f' \rangle_{\Gamma} \]

Sum over \( s \in S \): \( 0 = (\geq \text{LHS})' + (\leq \text{RHS})' + (\text{smaller})' \).

\[ \sum_{s} \langle \partial_s f | \partial_s f \rangle = \sum_{s} \langle \partial_s f | \partial_s f \rangle = 2 \langle \Delta f | f \psi^2 \rangle = 2(0 | f \psi^2) = 0. \]

\[ \| f' \|_{2,B(2R)} = \langle f^2 - f \psi \rangle_{B(2R)} \]

\[ \leq \langle f^2 \rangle_{B(2R)} + \langle f^2 \rangle_{B(2R)} \]

\[ \leq \| f \|_{2,B(2R+1)}^2 + \| f \|_{2,B(2R)}^2 \]

\[ \leq 2 \| f \|_{2,B(2R+1)}^2 \]

Schwarz Inequality: \[ \langle f \psi \rangle \leq \| f \|_2 \| \psi \|_2 \]

Step 2 (Reverse Poincaré Inequality)

\[ \sum_{s \in S} \| \partial_s f \|_{2,B(R)}^2 \leq \frac{C_2}{R^2} \| f \|_{2,B(3r)}^2 \quad \text{if } f \text{ harmonic.} \]

Proof.

\[ \text{Product rule} \quad \langle f \psi \rangle = \langle f' \psi' \rangle = \langle f^2 \psi \rangle \]

\[ \Rightarrow \langle f \psi \rangle = \langle f' \psi' \rangle + \langle f^2 \psi \rangle \]

\[ = \langle f' \psi' \rangle + \langle f^2 \psi \rangle \]

\[ \leq \langle f' \psi' \rangle \leq \frac{C}{R^2} \langle f' \psi \rangle_{B(2R)} + \frac{C}{R} \langle f' \psi \rangle_{\Gamma} \]

\[ \langle f' \psi \rangle = \sum_{x \in X} \langle f' \psi \rangle(x) \psi(x) \]

\[ \langle f' \psi \rangle_{\Gamma} = \| f' \psi \|_{2,R}^2 + \frac{C}{R^2} \langle f' \psi \rangle_{B(2R)} + \frac{C}{R} \langle f' \psi \rangle_{\Gamma} \]

Reverse P I: \[ \sum_{s \in S} \| \partial_s f \|_{2,B(R)}^2 \leq \frac{C}{R^2} \| f \|_{2,B(3r)}^2 \]

\[ \langle f' \psi \rangle_{\Gamma} = \| f' \psi \|_{2,R}^2 + \frac{C}{R^2} \langle f' \psi \rangle_{B(2R)} + \frac{C}{R} \langle f' \psi \rangle_{\Gamma} \]

Sum over \( s \in S \): \( 0 = (\geq \text{LHS})' + (\leq \text{RHS})' + (\text{smaller})' \).

\[ \frac{C}{R} \langle f' \psi \rangle_{\Gamma} = \langle f' \psi \rangle_{\Gamma} \leq \langle f' \psi \rangle_{\Gamma} \]

\[ \leq \| f' \psi \|_{2,R}^2 + \frac{C}{R^2} \langle f' \psi \rangle_{B(2R)}^2 + \frac{C}{R} \langle f' \psi \rangle_{\Gamma}^2 \leq \| f' \psi \|_{2,R}^2 + \frac{C}{R^2} \langle f' \psi \rangle_{B(2R)}^2 \]

Expository

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