Geometric interpretation of the $\mathbb{Q}$-rank of a locally symmetric space

Dave (Witte) Morris

Dept of Math and Comp Sci
University of Lethbridge
Lethbridge, AB, T1K 3M4, Canada

Dave.Morris@uleth.ca

http://people.uleth.ca/~dave.morris/

joint work with
Pralay Chatterjee
Rice University

short preprint available soon
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Abstract

Let $X = \Gamma\backslash G/K$ be a locally symmetric space of finite volume (and assume G is semisimple). The $\mathbb{Q}$-rank of X is defined from algebraic properties of the discrete group Γ, but it has geometric interpretations. In particular, the $\mathbb{Q}$-rank should be equal to the maximum dimension of a closed, simply connected flat in X. (A *flat* in a Riemannian manifold is a totally geodesic, flat submanifold.) This would answer of question of G. Tomanov and B. Weiss, and is analogous to the fact that if $\widetilde{X} = G/K$ is a symmetric space, then the $\mathbb{R}$-rank of G is equal to the maximum dimension of a closed, simply connected flat in $\widetilde{X}$. We discuss $\mathbb{Q}$-rank and progress toward a proof of this conjecture.

1. What is $\mathbb{Q}$-rank?

Recall. symmetric space $\widetilde{X}$: nice Riem mfld
(connected, homogeneous)
homogeneous: $\widetilde{X} = G/K$

Eg.
- $S^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$
- $\mathbb{H}^n = G/K$, $G \approx \mathrm{SO}(1,n)$ and $K = \mathrm{SO}(n)$
- G/K where $G = \mathrm{SL}(n,\mathbb{R})$ and $K = \mathrm{SO}(n)$

Recall. locally symmetric X: $\quad X = \Gamma\backslash\widetilde{X}$
(universal cover $\widetilde{X}$ is symmetric space)

Assumption. X has finite volume.

Eg. X = hyperbolic n-mfld $\Gamma\backslash\mathbb{H}^n$

Rem. We assume $\widetilde{X}$ has no Euclidean factors.
I.e., G is "semisimple."

Defn. rank $\widetilde{X}$ = max dim of flat in $\widetilde{X}$
flat $\cong \mathbb{R}^r \times \mathbb{T}^s$ (totally geodesic)

Defn. $\mathbb{R}$-rank $\widetilde{X}$ = max dim of (closed) flat $\cong \mathbb{R}^r$

Thm. $\mathbb{R}$-rank $\widetilde{X} > 0 \Leftrightarrow \widetilde{X}$ *is not compact.*

Defn. flatrank X = max dim of (closed) flat $\cong \mathbb{R}^r$
(*not a standard definition*)

Defn. $\mathbb{Q}$-rank $X = \cdots$ (structure of Γ)

Prop. $\mathbb{Q}$-rank X = *min* r*, s.t.*
X within bdd dist of finite union of r-flats

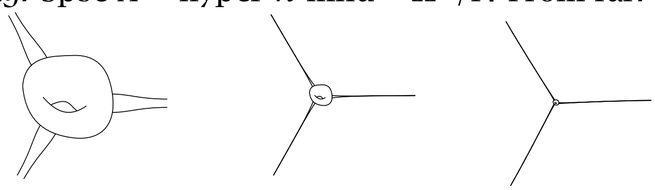
Note.
$\mathbb{Q}$-rank $X \le$ flatrank $X \le \mathbb{R}$-rank $\widetilde{X} \le$ rank $\widetilde{X}$.

Prop. $\mathbb{Q}\text{-rank}\, X = \min r$, *s.t.*
X within bdd dist of finite union of r-flats

Eg. Spse X is compact. Look at it from far away.

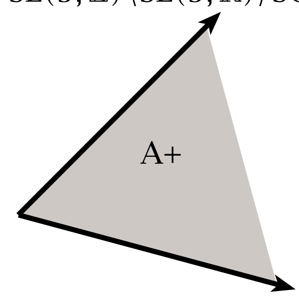
Limit = point = 0-dim'l simplicial cplx Σ^0.
$\mathbb{Q}\text{-rank}\, X = 0$.

Eg. Spse X = hyper n-mfld = $\mathbb{H}^n/\Gamma$. From far:



Limit = pencil of rays
= 1-dim'l simplicial cplx Σ^1 (= $\text{cone}(\Sigma^0)$).
$\mathbb{Q}\text{-rank}\, X = 1$.

Eg. Spse $X = \text{SL}(3,\mathbb{Z})\backslash\text{SL}(3,\mathbb{R})/\text{SO}(3)$. Far:



Limit = sector of $\mathbb{R}^2$
= 2-dim'l simp cplx Σ^2 (= $\text{cone}(\Sigma^1)$).
$\mathbb{Q}\text{-rank}\, X = 2$.

Thm (Hattori, Leuzinger). *X locally symmetric. Look from far away.*

- *Limit is a simplicial cplx $\Sigma^k = \text{cone}(\Sigma^{k-1})$.*
- *$k = \mathbb{Q}\text{-rank}\, X$.*

2. The conjecture

Defn. flatrank X = max dim of (closed) flat $\cong \mathbb{R}^r$

Prop. $\mathbb{Q}\text{-rank}\, X = \min r$, *s.t.*
X within bdd dist of finite union of r-flats

$\mathbb{Q}\text{-rank}\, X \le \text{flatrank}\, X \le \mathbb{R}\text{-rank}\, \widetilde{X} \le \text{rank}\, \widetilde{X}$.

Conj. $\mathbb{Q}\text{-rank}\, X = \text{flatrank}\, X$.

Thm (Tomanov-Weiss). $\mathbb{Q}\text{-rank}\, X < \mathbb{R}\text{-rank}\, \widetilde{X} \Rightarrow$

$\text{flatrank}\, X < \mathbb{R}\text{-rank}\, \widetilde{X}$.

Rem. Conjecture is:

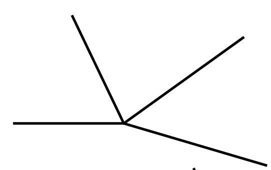
- trivial if $\mathbb{Q}\text{-rank}\, X = 0$;
- easy if $\mathbb{Q}\text{-rank}\, X = 1$.

Thm (Chatterjee-Morris). *Conjecture is true if $\mathbb{Q}\text{-rank}\, X = 2$.*

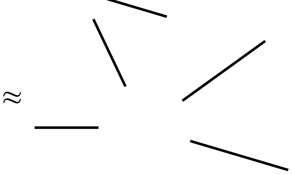
I.e., $\mathbb{Q}\text{-rank}\, X = 2 \Rightarrow \text{flatrank}\, X \le 2$.

3. A proof: $\mathbb{Q}\text{-rank}\, X = 1 \Rightarrow \text{flatrank}\, X \le 1$.

Spse $F \cong \mathbb{R}^2 \hookrightarrow X \approx$

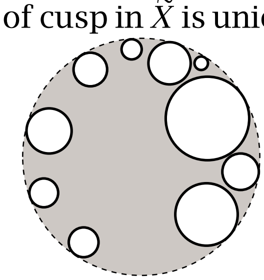


Large circle $S \hookrightarrow X \smallsetminus \text{cpct} \approx$



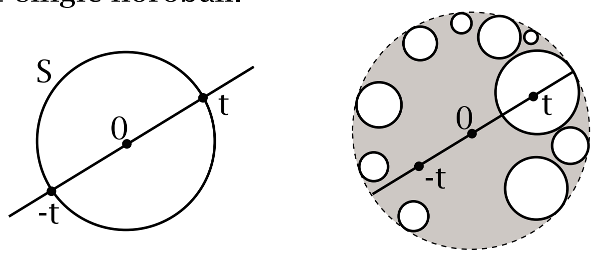
S connected $\Rightarrow S \subset$ single cusp.

Inverse image of cusp in $\widetilde{X}$ is union of horoballs:



S connected $\Rightarrow S \subset$ single horoball.

$S \subset$ single horoball.



γ_t and γ_{-t} are not in the same horoball. $\to\leftarrow$

In higher $\mathbb{Q}$-rank: $\gamma_t, \gamma_{-t} \in$ "same horoball"?

$\overline{\pi}: \widetilde{X} \smallsetminus \text{cpct} \to X \smallsetminus \text{cpct} \approx \Sigma^0 \times \mathbb{R} \to \Sigma^0$

$\overline{\pi}^{-1}(\text{simplex})$ = union of horoballs
$\Rightarrow$ horoball = connected component

Need: $(*)$ $f: S^r \to \Sigma^{r-1} \Rightarrow \exists z, \Delta,$
$z, -z$ in same conn comp of $\overline{\pi}^{-1}(\Delta)$.

Suffice: $\hat{f}: S^r \to \hat{\Sigma}^{r-1} \Rightarrow \exists z,\ f(z) = f(-z)$.
"antipodal coincidence"

4. Antipodal coincidence

Want: $\hat{f}: S^r \to \hat{\Sigma}^{r-1} \Rightarrow \exists z,\ f(z) = f(-z)$.
$r = \mathbb{Q}\text{-rank}\, X$

Thm (Borsuk-Ulam).
$f: S^n \to \mathbb{R}^n \Rightarrow$ *antipodal coincidence.*

Thm (Conner-Floyd).
$f: S^n \to M^n$ *not onto* $\Rightarrow$ *antipodal coincidence.*

Prop. $f: S^{2n+1} \to \Sigma^n \Rightarrow$ *antipodal coincidence.*

Proof. Σ^n embeds in $\mathbb{R}^{2n+1}$.

Cor. $\text{flatrank}\, X \le 2\ \mathbb{Q}\text{-rank}\, X - 1$.

Prop. $f: S^2 \to \Sigma^1 \Rightarrow$ *antipodal coincidence.*
This proves conjecture for $\mathbb{Q}\text{-rank}\, X = 2$.

Thm (Ščepin, 1974).
$\exists S^{2n-1} \to \Sigma^n$, *no antipodal coincidence.*

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