

**DANI'S CONTRIBUTIONS  
TO ERGODIC THEORY ON HOMOGENEOUS SPACES**

**A LECTURE AT THE CONFERENCE IN HONOR OF  
PROFESSOR S. G. DANI'S 60TH BIRTHDAY**

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*to Professor S. G. Dani on his 60th birthday*

Dani has published about 90 papers so far. Roughly 2/3 (more than 60) deal with dynamics on homogeneous spaces.

**The set up.**

- $G = \mathrm{SL}(n, \mathbb{R})$  (or, for the experts,  $G$  may be any connected Lie group),
- $g \in G$ ,
- $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ , or any other *lattice* in  $G$ :  $\Gamma$  is discrete and  $G/\Gamma$  has finite volume.

We study the iterates of  $g$  acting by translation on  $G/\Gamma$ .

That is, look at the maps  $g^n: x\Gamma \mapsto g^n x\Gamma$ .

Sometimes we replace  $\{g^n\}$  with one-parameter subgroup  $g^t = \exp(tv)$ ,  
or some other subgroup  $H$  of  $G$ .

**Quick list of topics.**

- (1) dynamics of general homogeneous flows
- (2) unipotent dynamics
- (3) applications in Number Theory
- (4) bounded orbits and Schmidt's Game
- (5) actions of lattices
- (6) actions on groups
- (7) finitely additive measures
- (8) Borel Density Theorem
- (9) survey/expository
- (10) etc.

Many of these contributions have been mentioned in other talks at this conference. We will discuss a few results from (1), (2), and (3).

## 1. DYNAMICS OF GENERAL HOMOGENEOUS FLOWS

We will quickly state a few questions that Dani worked on, but remain open.

§1A. **Kolmogorov automorphisms.** It is conjectured that translations with no 0-entropy quotients should be Bernoulli.

**Conjecture.** *Suppose*

- $G = \mathrm{SL}(n, \mathbb{R})$  (for simplicity)
- $g$  has an eigenvalue  $\lambda$ , such that  $|\lambda| \neq 1$  (i.e., entropy  $\neq 0$ )

*Then the action of  $g$  on  $G/\Gamma$  is measurably isomorphic to a Bernoulli shift.*

Dani (1976):

- $(g, G/\Gamma)$  has no 0-entropy quotients (i.e., it is a  $K$ -automorphism).
- The conjecture is true if  $g$  is diagonalizable (over  $\mathbb{C}$ ).

Open case:  $g$  has nontrivial Jordan blocks with eigenvalues on the unit circle.

§1B. **Anosov diffeomorphisms.**

**Definition.** A diffeomorphism  $f$  of a compact manifold  $M$  is *Anosov* if, at every point  $p \in M$ , the tangent space  $T_p M$  has a splitting  $T_p M = \mathcal{E}^+ \oplus \mathcal{E}^-$ , such that

- for  $v \in \mathcal{E}^+$ ,  $D(f^{-n})(v) \rightarrow 0$  exponentially fast as  $n \rightarrow \infty$
- for  $v \in \mathcal{E}^-$ ,  $D(f^n)(v) \rightarrow 0$  exponentially fast as  $n \rightarrow \infty$

**Conjecture** (60's). *If there is an Anosov diffeomorphism on  $M$ , then  $M$  is finitely covered by a nilmanifold  $G/\Gamma$ , where  $G$  is nilpotent.*

*(and the diffeomorphism lifts to an affine map)*

Dani:

- settled some cases where  $M$  is a double-coset space  $K \backslash G/H$  (1980)
- constructed nilmanifolds (for example, of every sufficiently large dimension) that do have Anosov diffeomorphisms (1978, with Mainkar 2005)

2. APPLICATIONS IN NUMBER THEORY

Unipotent dynamics (which will be discussed later in the lecture) became famous because of the following application, known as:

- Margulis' Theorem on Values of Quadratic Forms, or
- the Oppenheim Conjecture.

**Theorem** (Margulis 1987). *Suppose*

- $Q(x_1, x_2, \dots, x_n)$  is a real quadratic form  
(i.e.,  $Q$  is a homogeneous polynomial of degree 2, with real coefficients),
- $Q$  is **not** a scalar multiple of a quadratic form with integer coefficients,
- $Q$  is indefinite  
(i.e.,  $Q(\mathbb{R}^n)$  contains both some positive values and some negative values),
- $n \geq 3$ , and
- $Q$  is nondegenerate  
(i.e., no linear change of variables transforms  $Q$  into a form with  $< n$  variables).

Then  $Q(\mathbb{Z}^n)$  is dense in  $\mathbb{R}$ .

**Remark.** The hypotheses cannot be eliminated:

- If  $Q \in \mathbb{Z}[x_1, \dots, x_n]$ , then  $Q(\mathbb{Z}^n) \subset \mathbb{Z}$  is discrete, so it is not dense in  $\mathbb{R}$ . Multiplying  $Q$  by a scalar cannot make the range of  $Q$  dense.
- If  $Q$  fails to be indefinite, then  $Q(\mathbb{Z}^n) \subset Q(\mathbb{R}^n)$  is contained in a half-line, so it is not dense.
- Let  $Q(x_1, x_2) = x_1^2 - \alpha^2 x_2^2$ . It is not difficult to see that if  $\alpha$  is badly approximable by rationals, then 0 is *not* an accumulation point of  $Q(\mathbb{Z}^2)$ . Obviously, then  $Q(\mathbb{Z}^2)$  is not dense in  $\mathbb{R}$ .
- Given a 2-variable counterexample one can easily construct degenerate counterexamples in any number of variables: e.g.,  $Q(x_1, x_2, \dots, x_n) = (x_1 + \dots + x_{n-1})^2 - \alpha^2 x_n^2$ .

**Example.** If  $Q(x, y, z) = x^2 - \sqrt{2}y^2 + \pi z^2$ , then  $Q(\mathbb{Z}^3)$  is dense in  $\mathbb{R}$ .

*Idea of proof.* Although it was originally proved independently, the theorem can now be obtained as a corollary of Ratner's Theorem on orbit closures.

- Given  $a \in \mathbb{R}$ .
- Assume, for simplicity, that  $n = 3$ .
- $Q$  indefinite  $\implies \exists \vec{v} \in \mathbb{R}^3, Q(\vec{v}) = a$ .
- Let  $H = \text{SO}_3(Q) \subset \text{SL}(n, \mathbb{R}) = G, \quad \Gamma = \text{SL}(n, \mathbb{Z})$ .
- The irrationality assumption on  $Q(\vec{x})$  implies the  $H$ -orbit of  $e\Gamma$  is dense in  $G/\Gamma$   
[This is a consequence of Ratner's Theorem.]  
I.e.,  $H\Gamma$  is dense in  $G$ .
- Since  $G$  is transitive on the nonzero vectors in  $\mathbb{R}^3$ , this implies  $\exists h \in H$  and  $\gamma \in \Gamma$ ,  
such that  $h\gamma\vec{e}_1 \approx \vec{v}$ , where  $\vec{e}_1 = (1, 0, 0)$ .
- Let  $\vec{m} = \gamma\vec{e}_1 \in \mathbb{Z}^3$ .
- Then  $Q(\vec{m}) = Q(\gamma\vec{e}_1) = Q(h\gamma\vec{e}_1) \approx Q(\vec{v}) = a.$  □

Dani and Margulis made important improvements, including:

- (1989) approximation by primitive vectors:  $Q(\vec{m}) \approx a$  with  $\vec{m}$  primitive
- (1990) simultaneous approximation by two forms:  $Q_1(\vec{m}) \approx a_1$  and  $Q_2(\vec{m}) \approx a_2$

- (1993) quantitative lower bound on the number of solutions of  $Q(\vec{m}) \in [a, b]$   
— the first use of *linearization*

This is a forerunner of *many* results on counting lattice points in varieties  
(Eskin, Margulis, Mozes, Oh, Shah, Lindenstrauss, ...)

More recently (Dani-Nogueira 2002): e.g.,  $Q(x, y) = (x - \varphi y)(x - \psi y)$ ,

- $\varphi$  irrational number such that every finite sequence of positive integers appears in the continued fraction expansion of  $\varphi$ , and
- $\psi$  is any real number other than  $\psi$ .

Then, for any  $v \in \mathbb{N}^2$ ,  $Q(\mathrm{SL}(2, \mathbb{Z})^+v)$  is dense in  $\mathbb{R}$ .

## 3. UNIPOTENT DYNAMICS

**Definition.** one-parameter subgroup  $\{u^t\} \subset \mathrm{SL}(n, \mathbb{R})$  is *unipotent* if  $\{u^t\} \subset \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}$  (after a change of basis). I.e., 1 is the only eigenvalue of  $u^t$  (for all  $t$ ).

**Example.**

- $u^t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$  in  $\mathrm{SL}(2, \mathbb{R})$
- $u^t = \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$  in  $\mathrm{SL}(3, \mathbb{R})$

In general the closure of an orbit in a dynamical system (even for  $g^t$  on  $G/\Gamma$ ) can be a bad set (e.g., a fractal). A fundamental result in the subject is that this does not happen in unipotent dynamics:

**Theorem** (Ratner,  $\sim 1990$ ). *Assume  $u^t$  is unipotent.*

- (C) *The closure of every  $u^t$ -invariant orbit on  $G/\Gamma$  is a (finite-volume, homogeneous) submanifold.*
- (M) *Every ergodic  $u^t$ -invariant probability measure on  $G/\Gamma$  is the natural Lebesgue measure on some (finite-volume, homogeneous) submanifold.*
- (E) *Every  $u^t$ -orbit is uniformly distributed in its closure.*

(C stands for “closure,” M stands for “measure,” E stands for “equidistribution.”)

Dani was a central figure in the activity that preceded this result. In particular:

**Theorem.**

- (C,M,E) for  $G = \mathrm{SL}(2, \mathbb{R})$ .  
[Hedlund 1936, Furstenberg 1973, Dani 1982, Dani-Smillie 1984]
- (C,M) if  $G$  semisimple and  $u^t$  replaced with maximal unipotent subgroup  $\begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}$ .  
(or, more generally, any “horospherical” subgroup)  
[Veech 1975, 1977; Dani 1978, 1980, 1986]
- (C) if  $G = \mathrm{SL}(3, \mathbb{R})$  and  $u^t$  is generic. [Dani-Margulis 1989, 1990]

Furthermore, for applications, it is important to have versions of (E) giving estimates that are *uniform* as the starting point of the orbit varies over a compact set. The first such theorem was proved by Dani and Margulis (1992, 1993).

**Remark.**

- (C) was conjectured by Raghunathan (unpublished 1975).  
Dani’s early work was one of the factors that led to the conjecture.
- (M) was conjectured by Dani (1981).

**Remark** (Greenleaf, 1963).

$G = \mathrm{SL}(n, \mathbb{R})$ ,  $U$  maximal unipotent subgroup,  $G/\Gamma$  compact

$\implies$  every  $U$ -orbit on  $G/\Gamma$  is dense

$\implies U g \Gamma$  is dense in  $G$ ,  $\forall g$

$\implies \Gamma g U$  is dense in  $G$ ,  $\forall g$

$\implies$  every  $\Gamma$ -orbit on  $G/U$  is dense.

Since  $U$  fixes the vector  $\vec{e}_1$ , then  $\Gamma g \vec{e}_1$  is dense in  $\mathbb{R}^n$ .

I.e., every  $\Gamma$ -orbit on  $\mathbb{R}^n$  is dense.

Dani (1973) proved a  $p$ -adic version of this. This work is interesting for several reasons:

- It was his first paper.
- It was joint with Mrs. Dani, while they were students here at TIFR.
- $\exists$  only one other paper where Dani's coauthor precedes him in alphabetical order: Benardete (1999) about flows on solvmanifolds.

Two results of Dani and Margulis have become cornerstones of the subject:

**Theorem** (Margulis 1972, Dani 1979, 1984, 1986, Dani-Margulis 1991).

*No  $u^t$ -orbit diverges to  $\infty$ . I.e., for  $x \in G/\Gamma$ ,  $d(u^t x, x) \not\rightarrow \infty$  as  $t \rightarrow +\infty$ .*

*In fact, no orbit spends a significant portion of its life far from home:*

$$\lim_{C \rightarrow \infty} \limsup_{T \rightarrow \infty} \frac{\#\{n \in [0, T] \mid d(u^n x, x) > C\}}{T} = 0.$$

*That is, when averaging over longer and longer intervals of the orbit, no mass escapes to  $\infty$ .*

*Proof for  $G = \mathrm{SL}(2, \mathbb{R})$ ,  $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ . Suppose  $d(u^{n_0} x \Gamma, x \Gamma)$  is large.*

*Mahler Compactness:  $\exists$  nonzero  $\vec{v} \in x\mathbb{Z}^2$ ,  $u^{n_0} \vec{v} \approx \vec{0}$ .*

*Choose  $n_- < n_0$ ,  $n_+ > n_0$ ,  $\|u^{n_\pm} \vec{v}\| \doteq 1$ .*

*For  $n \in [n_-, n_+] \setminus$  small interval,  $u^n \vec{v} \not\approx \vec{0}$ .*

*Indeed,  $\nexists \vec{w} \in x\mathbb{Z}^2$ ,  $u^{n_0} \vec{w} \approx \vec{0}$ .*

*Else,*

$$\mathrm{Area}(\Delta(\vec{0}, u^t \vec{v}, u^t \vec{w})) \leq \frac{1}{2} \|u^t \vec{v}\| \|u^t \vec{w}\| \approx 0.$$

*But any lattice triangle in  $\mathbb{Z}^2$  has area  $\geq 1/2$*

*$x \in \mathrm{SL}(2, \mathbb{R}) \implies x$  area-preserving  $\implies$  any lattice triangle in  $x\mathbb{Z}^2$  has area  $\geq 1/2$*

$\rightarrow \leftarrow$

Thus,  $d(u^n x \Gamma, x \Gamma)$  is not large, except in a small subinterval. □

**General case.**

- $u^t$  unipotent

$\implies u^t = \exp(tM)$  where  $M$  is nilpotent ( $M^n = 0$ )

$\implies$  matrix entries of  $u^t$  are polynomial functions (of degree  $\leq n$ )

$\implies \|u^t v\|^2$  is a polynomial  $f(x)$  (of degree  $\leq 2n$ )

$\implies \ell(\{x \mid |f(x)| < \epsilon\}) \ll \ell(\{x \mid \epsilon < |f(x)| < 1\})$ .

- But proof is much more complicated, because  $x\mathbb{Z}^n$  can have many small vectors.

E.g.,  $\langle \epsilon(1, 0, 0), \epsilon(0, 1, 0), (1/\epsilon^2)(0, 0, 1) \rangle$  in  $\mathbb{R}^3$ .

For nice exposition, see appendix of (Dani-Margulis, *Ens. Math.*, 1990).

**Theorem** (Dani-Margulis Linearization 1993; roots in Dani-Smillie 1984 and Shah 1991).

*A rigorous version of the following argument, and providing uniform estimates.*

*It replaces the action of  $u^t$  on  $G/\Gamma$  with the linear action of  $u^t$  on a vector space.*

*Idea of proof of equidistribution from classification of measures.*

The key is to show:

- Suppose  $M$  is a submanifold that supports an ergodic  $u^t$ -invariant measure.
- Then the orbit of  $g\Gamma$  spends  $< \epsilon\%$  of its life near  $M$   
(unless  $g\Gamma \in M$ ).

Linearization:  $d(u^t g\Gamma, M) \simeq$  polynomial of degree  $< N$ .

Thus, spends  $< \epsilon\%$  of its life less than  $\delta$ . □

**Linearization.**

Since  $M$  is homogeneous, it is an orbit of some connected subgroup  $H$  of  $G$ .

(To avoid complications, we assume  $H = N_G(H)$ .)

One can show that  $H$  is “Zariski closed” (since unipotents generate a cocompact subgroup).

So Chevalley’s Theorem (from theory of Algebraic Groups):

$$\exists G \hookrightarrow \mathrm{SL}(\ell, \mathbb{R}), \text{ such that } H = \mathrm{Stab}_G(v). \quad (\text{Namely, } v \in \bigwedge^k \mathfrak{h} \subset \bigwedge^k \mathfrak{g}.)$$

Therefore  $G/H \hookrightarrow \mathbb{R}^\ell: g\Gamma \mapsto gv = \underline{g}$ .

Furthermore,  $G/H$  is a *subvariety* – defined by polynomial equations:

$$G/H = \{ \vec{x} \in \mathbb{R}^\ell \mid f(\vec{x}) = 0 \}, \text{ where } f(\vec{x}) \text{ is a polynomial of degree } N.$$

Now  $d(\vec{x}, \vec{v})^2$  is a polynomial of degree 2,

so  $d(u^t \underline{g}, \vec{v})^2$  is a polynomial of degree  $\leq 2N$ .

Therefore, it is rarely small, so  $u^t g$  is rarely close to  $M$ . □